THE FIRST SIX BOOKS OF

THE ELEMENTS OF EUCLID

IN WHICH COLOURED DIAGRAMS AND SYMBOLS

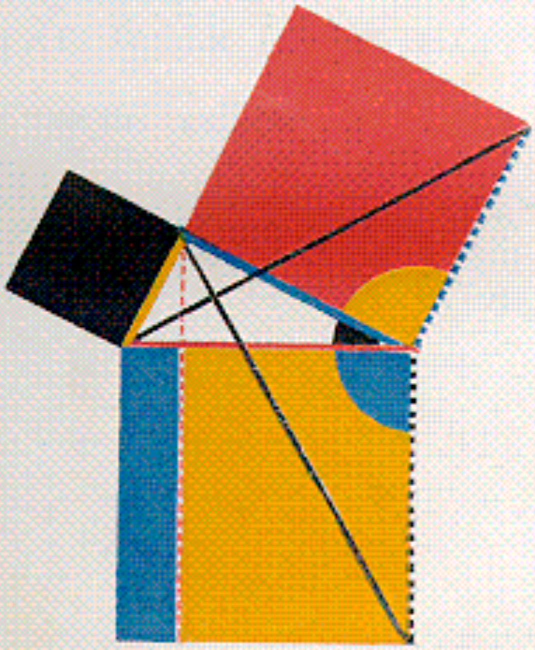
ARE USED INSTEAD OF LETTERS FOR THE

GREATER EASE OF LEARNERS

BY OLIVER BYRNE

SURVEYOR OF HER MAJESTY'S SETTLEMENTS IN THE FALKLAND ISLANDS

AND AUTHOR OF NUMEROUS MATHEMATICAL WORKS



LONDON

WILLIAM PICKERING

1847

TO THE

RIGHT HONOURABLE THE EARL FITZWILLIAM,

ETC. ETC. ETC.

THIS WORK IS DEDICATED

BY HIS LORDSHIP'S OBEDIENT

AND MUCH OBLIGED SERVANT,

OLIVER BYRNE.

BYRNE'S EUCLID

THE FIRST SIX BOOKS OF

THE ELEMENTS OF EUCLID

WITH COLOURED DIAGRAMS

AND SYMBOLS

INTRODUCTION.

THE arts and ſciences have become ſo extenſive, that to facilitate their acquirement is of as much importance as to extend their boundaries. Illuſtration, if it does not ſhorten the time of ſtudy, will at leaſt make it more agreeable. THIS WORK has a greater aim than mere illuſtration; we do not introduce colours for the purpoſe of entertainment, or to amuſe *by certain combinations of tint and form*, but to aſſiſt the mind in its reſearches after truth, to increaſe the facilities of inſtruction, and to diffuſe permanent knowledge. If we wanted authorities to prove the importance and uſefulneſs of geometry, we might quote every philoſopher ſince the days of Plato. Among the Greeks, in ancient, as in the ſchool of Peſtalozzi and others in recent times, geometry was adopted as the beſt gymnaſtic of the mind. In fact, Euclid's Elements have become, by common conſent, the baſis of mathematical ſcience all over the civilized globe. But this will not appear extraordinary, if we conſider that this ſublime ſcience is not only better calculated than any other to call forth the ſpirit of inquiry, to elevate the mind, and to ſtrengthen the reaſoning faculties, but alſo it forms the beſt introduction to moſt of the uſeful and important vocations of human life. Arithmetic, land-ſurveying, menſuration, engineering, navigation, mechanics, hydroſtatics, pneumatics, optics, phyſical aſtronomy, &c. are all dependent on the propoſitions of geometry.

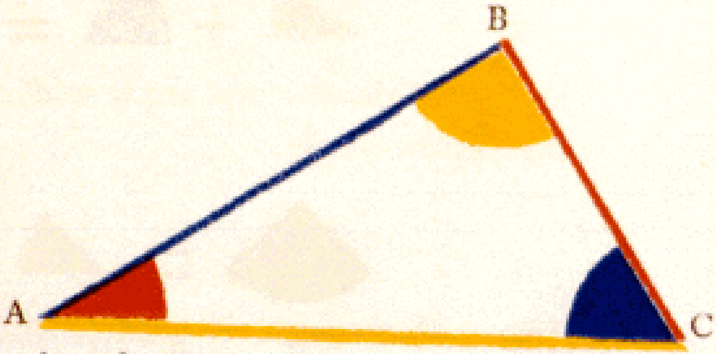
Much however depends on the firſt communication of any ſcience to a learner, though the beſt and moſt eaſy methods are ſeldom adopted. Propoſitions are placed before a ſtudent, who though having a ſufficient underſtanding, is told juſt as much about them on entering at the very threſhold of the ſcience, as gives him a prepoſſeſſion moſt unfavourable to his future ſtudy of this delightful ſubject; or "the formalities and paraphernalia of rigour are ſo oſtentatiouſly put forward, as almoſt to hide the reality. Endleſs and perplexing repetitions, which do not confer greater exactitude on the reaſoning, render the demonſtrations involved and obſcure, and conceal from the view of the ſtudent the conſecution of evidence." Thus an averſion is created in the mind of the pupil, and a ſubject ſo calculated to improve the reaſoning powers, and give the habit of cloſe thinking, is degraded by a dry and rigid courſe of inſtruction into an unintereſting exerciſe of the memory. To raiſe the curioſity, and to awaken the liſtleſs and dormant powers of younger minds ſhould be the aim of every teacher; but where examples of excellence are wanting, the attempts to attain it are but few, while eminence excites attention and produces imitation. The object of this Work is to introduce a method of teaching geometry, which has been much approved of by many ſcientific men in this country, as well as in France and America. The plan here adopted forcibly appeals to the eye, the moſt ſenſitive and the moſt comprehenſive of our external organs, and its pre-eminence to imprint it ſubject on the mind is ſupported by the incontrovertible maxim expreſſed in the well known words of Horace:—

Segnius irritant animos demiſſa per aurem
Quàm quæ ſunt oculis ſubjecta fidelibus.

A feebler impreſs through the ear is made,
Than what is by the faithful eye conveyed.

All language consists of representative signs, and those signs are the best which effect their purposes with the greatest precision and dispatch. Such for all common purposes are the audible signs called words, which are still considered as audible, whether addressed immediately to the ear, or through the medium of letters to the eye. Geometrical diagrams are not signs, but the materials of geometrical science, the object of which is to show the relative quantities of their parts by a process of reasoning called Demonstration. This reasoning has been generally carried on by words, letters, and black or uncoloured diagrams; but as the use of coloured symbols, signs, and diagrams in the linear arts and sciences, renders the process of reasoning more precise, and the attainment more expeditious, they have been in this instance accordingly adopted.

Such is the expedition of this enticing mode of communicating knowledge, that the Elements of Euclid can be acquired in less than one third the time usually employed, and the retention by the memory is much more permanent; these facts have been ascertained by numerous experiments made by the inventor, and several others who have adopted his plans. The particulars of which are few and obvious; the letters annexed to points, lines, or other parts of a diagram are in fact but arbitrary names, and represent them in the demonstration; instead of these, the parts being differently coloured, are made to name themselves, for their forms in corresponding colours represent them in the demonstration.



In order to give a better idea of this system, and of the advantages gained by its adoption, let us take a right angled triangle, and expreſs ſome of its properties both by colours and the method generally employed.

Some of the properties of the right angled triangle ABC, *expreſſed by the method generally employed.*

1. The angle BAC, together with the angles BCA and ABC are equal to two right angles, or twice the angle ABC.

2. The angle CAB added to the angle ACB will be equal to the angle ABC.

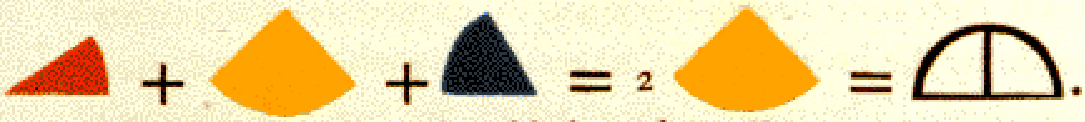
3. The angle ABC is greater than either of the angles BAC or BCA.

4. The angle BCA or the angle CAB is leſs than the angle ABC.

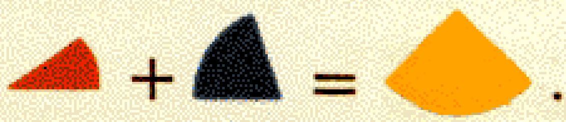
5. If from the angle ABC, there be taken the angle BAC, the remainder will be equal to the angle ACB.

6. The ſquare of AC is equal to the ſum of the ſquares of AB and BC.

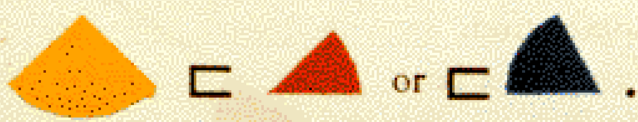
The ſame properties expreſſed by colouring the different parts.

1. 

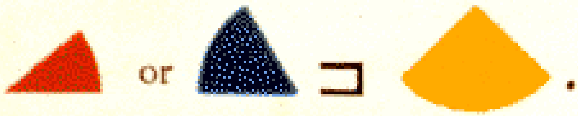
That is, the red angle added to the yellow angle added to the blue angle, equal twice the yellow angle, equal two right angles.

2. 

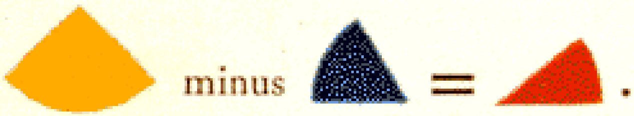
Or in words, the red angle added to the blue angle, equal the yellow angle.

3. 

The yellow angle is greater than either the red or blue angle.

4. 

Either the red or blue angle is leſs than the yellow angle.

5. 

In other terms, the yellow angle made leſs by the blue angle equal the red angle.

6. 

That is, the ſquare of the yellow line is equal to the ſum of the ſquares of the blue and red lines.

In oral demonſtrations we gain with colours this important advantage, the eye and the ear can be addreſſed at the ſame moment, ſo that for teaching geometry, and other linear arts and ſciences, in claſſes, the ſyſtem is the beſt ever propoſed, this is apparent from the examples juſt given.

Whence it is evident that a reference from the text to the diagram is more rapid and ſure, by giving the forms and colours of the parts, or by naming the parts and their colours, than naming the parts and letters on the diagram. Beſides the ſuperior ſimplicity, this ſyſtem is likewiſe conſpicuous for concentration, and wholly excludes the injurious though prevalent practice of allowing the ſtudent to commit the demonſtration to memory; until reaſon, and fact, and proof only make impreſſions on the underſtanding.

Again, when lecturing on the principles or properties of figures, if we mention the colour of the part or parts referred to, as in ſaying, the red angle, the blue line, or lines, &c. the part or parts thus named will be immediately ſeen by all in the claſs at the ſame inſtant; not ſo if we ſay the angle ABC, the triangle PFQ, the figure EGKt, and ſo on;

for the letters muſt be traced one by one before the ſtudents arrange in their minds the particular magnitude referred to, which often occaſions confuſion and error, as well as loſs of time. Alſo if the parts which are given as equal, have the ſame colours in any diagram, the mind will not wander from the object before it; that is, ſuch an arrangement preſents an ocular demonſtration of the parts to be proved equal, and the learner retains the data throughout the whole of the reaſoning. But whatever may be the advantages of the preſent plan, if it be not ſubſtituted for, it can always be made a powerful auxiliary to the other methods, for the purpoſe of introduction, or of a more ſpeedy reminiſcence, or of more permanent retention by the memory.

The experience of all who have formed ſyſtems to impreſs facts on the underſtanding, agree in proving that coloured repreſentations, as pictures, cuts, diagrams, &c. are more eaſily fixed in the mind than mere ſentences unmarked by any peculiarity. Curious as it may appear, poets ſeem to be aware of this fact more than mathematicians; many modern poets allude to this viſible ſyſtem of communicating knowledge, one of them has thus expreſſed himſelf:

> Sounds which addreſs the ear are loſt and die
> In one ſhort hour, but theſe which ſtrike the eye,
> Live long upon the mind, the faithful ſight
> Engraves the knowledge with a beam of light.

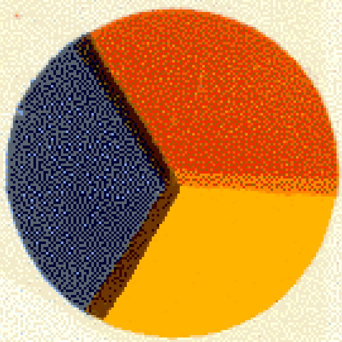
This perhaps may be reckoned the only improvement which plain geometry has received ſince the days of Euclid, and if there were any geometers of note before that time, Euclid's ſucceſs has quite eclipſed their memory, and even occaſioned all good things of that kind to be aſſigned to him; like Æſop among the writers of Fables. It may alſo be worthy of remark, as tangible diagrams afford the only medium through which geometry and other linear arts and ſciences can be taught to the blind, this viſible ſyſtem is no leſs adapted to the exigencies of the deaf and dumb.

Care muſt be taken to ſhow that colour has nothing to do with the lines, angles, or magnitudes, except merely to name them. A mathematical line, which is length without breadth, cannot poſſeſs colour, yet the junction of two colours on the ſame plane gives a good idea of what is meant by a mathematical line; recollect we are ſpeaking familiarly, ſuch a junction is to be underſtood and not the colour, when we ſay the black line, the red line or lines, &c.

Colours and coloured diagrams may at firſt appear a clumſy method to convey proper notions of the properties and parts of mathematical figures and magnitudes, however they will be found to afford a means more refined and extenſive than any that has been hitherto propoſed.

We ſhall here define a point, a line, and a ſurface, and demonſtrate a propoſition in order to ſhow the truth of this aſſertion.

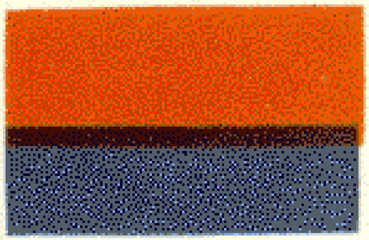
A point is that which has poſition, but not magnitude; or a point is poſition only, abſtracted from the conſideration of length, breadth, and thickneſs. Perhaps the following deſcription is better calculated to explain the nature of a mathematical point to thoſe who have not acquired the idea, than the above ſpecious definition.



Let three colours meet and cover a portion of the paper, where they meet is not blue, nor is it yellow, nor is it red, as it occupies no portion of the plane, for if it did, it would belong to the blue, the red, or the yellow part; yet it exiſts, and has poſition without magnitude, ſo that with a little reflection, this junction of three colours on a plane, gives a good idea of a mathematical point.

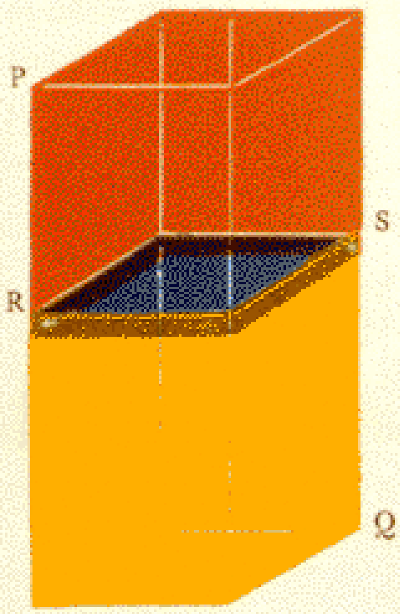
A line is length without breadth. With the aſſiſtance of colours, nearly in the ſame manner as before, an idea of a line may be thus given :—

Let two colours meet and cover a portion of the paper;



where they meet is not red, nor is it blue; therefore the junction occupies no portion of the plane, and therefore it cannot have breadth, but only length: from which we can readily form an idea of what is meant by a mathematical line. For the purpoſe of illuſtration, one colour differing from the colour of the paper, or plane upon which it is drawn, would have been ſufficient; hence in future, if we ſay the red line, the blue line, or lines, &c. it is the junctions with the plane upon which they are drawn are to be underſtood.

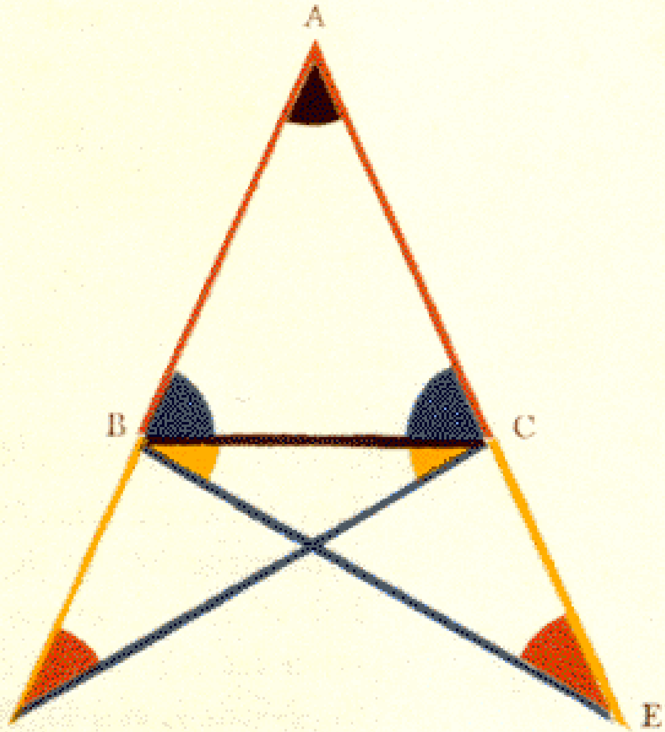
Surface is that which has length and breadth without thickneſs.



When we conſider a ſolid body (PQ), we perceive at once that it has three dimenſions, namely :— length, breadth, and thickneſs; ſuppoſe one part of this ſolid (PS) to be red, and the other part (QR) yellow, and that the colours be diſtinct without commingling, the blue ſurface (RS) which ſeparates theſe parts, or which is the ſame thing, that which divides the ſolid without loſs of material, muſt be without thickneſs, and only poſſeſſes length and breadth;

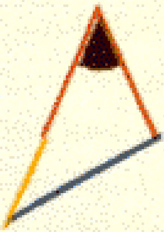
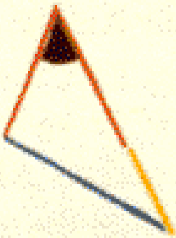
this plainly appears from reaſoning, ſimilar to that juſt employed in defining, or rather deſcribing a point and a line.

The propoſition which we have ſelected to elucidate the manner in which the principles are applied, is the fifth of the firſt Book.



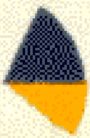
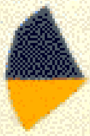
In an iſoſceles triangle ABC, the internal angles at the baſe ABC, ACB are equal, and when the ſides AB, AC are produced, the external angles at the baſe BCE, CBD are alſo equal.

Produce and
make =
Draw and 
(B. 1. pr. 3.)

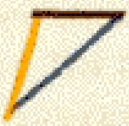
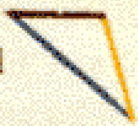
in  and 

we have =

 = and  common :

∴  = ,  = 

and  =  (B. 1. pr. 4.)

Again in  and .

 = ,

,

and ;

∴ 

and  (B. 1. pr. 4).

But ,

∴ .

Q. E. D.

By annexing Letters to the Diagram.

LET the equal ſides AB and AC be produced through the extremities BC, of the third ſide, and in the produced part BD of either, let any point D be aſſumed, and from the other let AE be cut off equal to AD (B. 1. pr. 3). Let the points E and D, ſo taken in the produced ſides, be connected by ſtraight lines DC and BE with the alternate extremities of the third ſide of the triangle.

In the triangles DAC and EAB the ſides DA and AC are reſpectively equal to EA and AB, and the included angle A is common to both triangles. Hence (B. 1. pr. 4.) the line DC is equal to BE, the angle ADC to the angle AEB, and the angle ACD to the angle ABE; if from the equal lines AD and AE the equal ſides AB and AC be taken, the remainders BD and CE will be equal. Hence in the triangles BDC and CEB, the ſides BD and DC are reſpectively equal to CE and EB, and the angles D and E included by thoſe ſides are alſo equal. Hence (B. 1. pr. 4.)

the angles DBC and ECB, which are thofe included by the third fide BC and the productions of the equal fides AB and AC are equal. Alfo the angles DCB and EBC are equal if thofe equals be taken from the angles DCA and EBA before proved equal, the remainders, which are the angles ABC and ACB oppofite to the equal fides, will be equal.

Therefore in an ifofceles triangle, &c.

Q. E. D.

Our object in this place being to introduce the fyftem rather than to teach any particular fet of propofitions, we have therefore felected the foregoing out of the regular courfe. For fchools and other public places of inftruction, dyed chalks will anfwer to defcribe diagrams, &c. for private ufe coloured pencils will be found very convenient.

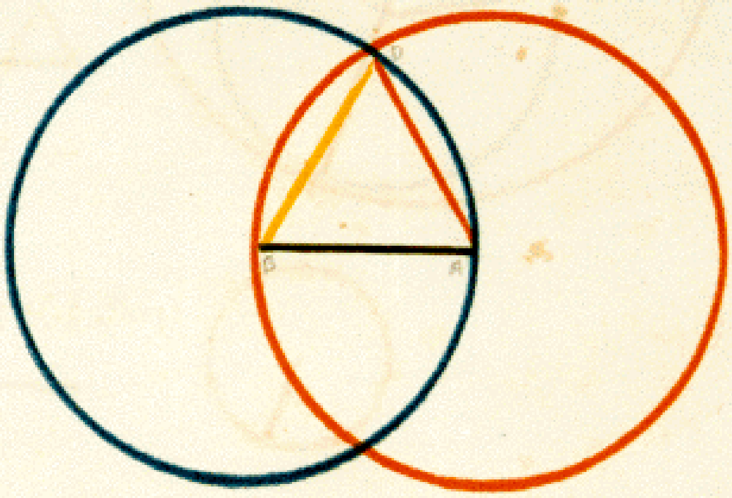
We are happy to find that the Elements of Mathematics now forms a confiderable part of every found female education, therefore we call the attention of thofe interefted or engaged in the education of ladies to this very attractive mode of communicating knowledge, and to the fucceeding work for its future developement.

We fhall for the prefent conclude by obferving, as the fenfes of fight and hearing can be fo forcibly and inftantaneously addreffed alike with one thoufand as with one, *the million* might be taught geometry and other branches of mathematics with great eafe, this would advance the purpofe of education more than any thing that *might* be named, for it would teach the people how to think, and not what to think; it is in this particular the great error of education originates.

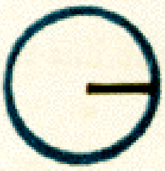
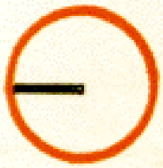
Euclid.

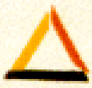
BOOK I.

PROPOSITION I. PROBLEM.



ON *a given finite straight line* (——) *to describe an equilateral triangle.*

Describe  and  (postulate 3.); draw —— and —— (post. 1.).

then will  be equilateral.

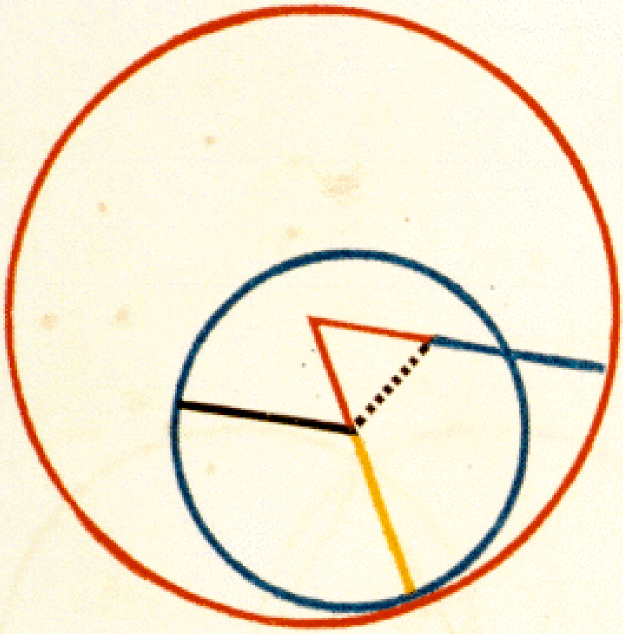
For —— = —— (def. 15.);

and —— = —— (def. 15.),

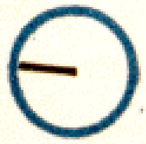
∴ —— = —— (axiom. 1.);

and therefore  is the equilateral triangle required.

Q. E. D.

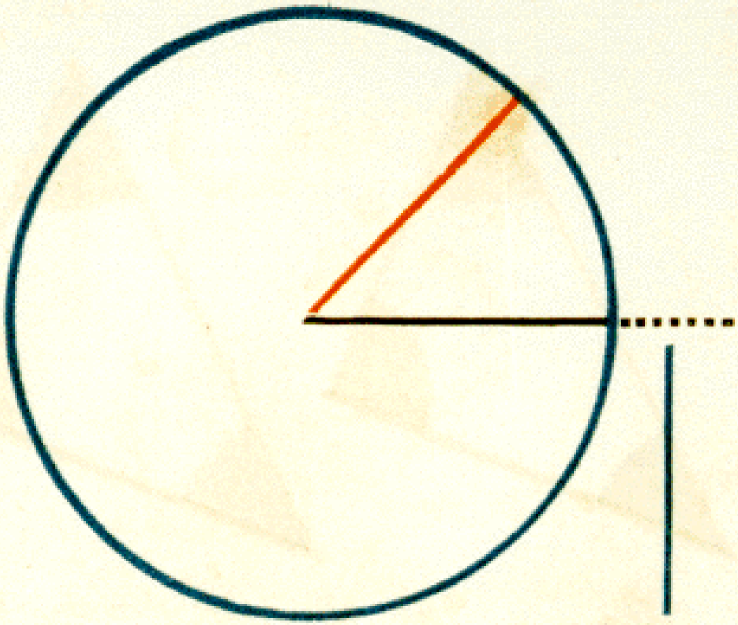


FROM *a given point* ( ), *to draw a ſtraight line equal to a given finite ſtraight line* ( ).

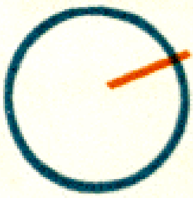
Draw (poſt. 1.), deſcribe  (pr. 1.), produce (poſt. 2.), deſcribe  (poſt. 3.), and  (poſt. 3.); produce (poſt. 2.), then is the line required.

For = (def. 15.), and = (conſt.), ∴ = (ax. 3.), but (def. 15.) = = ; ∴ drawn from the given point ( ), is equal the given line .

Q. E. D.



FROM *the greater* () *of two given straight lines, to cut off a part equal to the less* ().

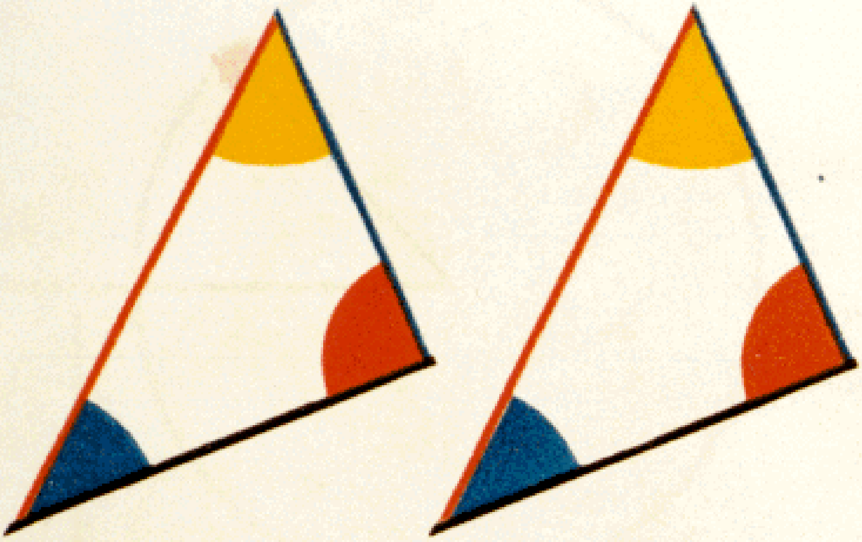
Draw = (pr. 2.); describe  (post. 3.), then = .

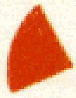
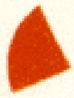
For = (def. 15.),

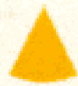
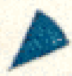
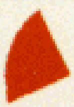
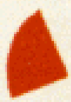
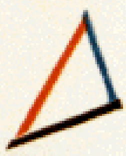
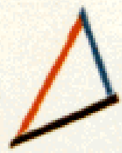
and = (const.);

∴ = (ax. 1.).

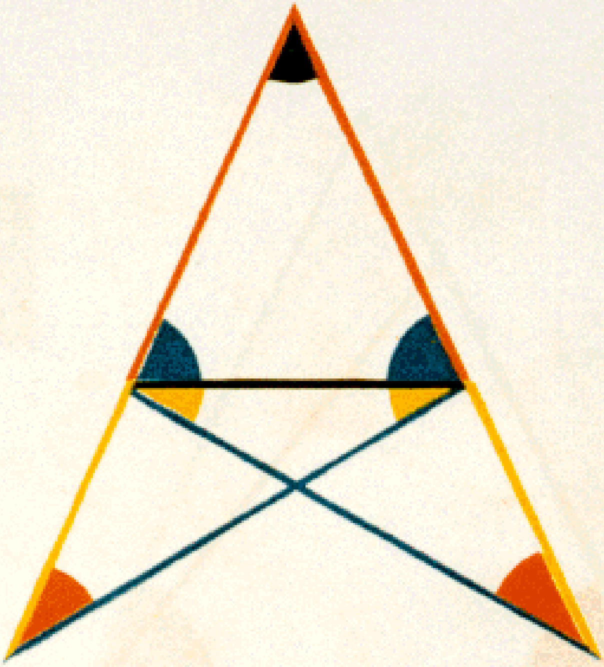
Q. E. D.

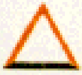


IF two triangles have two sides of the one respectively equal to two sides of the other, (to and to) and the angles ( *and*  *) contained by those equal sides also equal; then their bases or their sides (and) are also equal: and the remaining and their remaining angles opposite to equal sides are respectively equal (*  *=*  *and*  *=*  *): and the triangles are equal in every respect.*

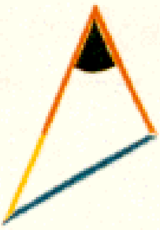
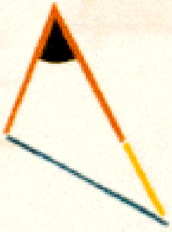
Let the two triangles be conceived, to be so placed, that the vertex of the one of the equal angles,  or  ; shall fall upon that of the other, and to coincide with , then will coincide with if applied: consequently will coincide with , or two straight lines will enclose a space, which is impossible (ax. 10), therefore = ,  =  and  = , and as the triangles  and  coincide, when applied, they are equal in every respect.

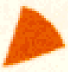
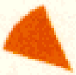
Q. E. D.

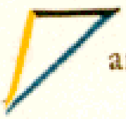
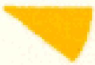
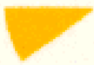
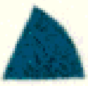
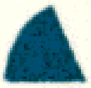


In any isosceles triangle  *if the equal sides be produced, the external angles at the base are equal, and the internal angles at the base are also equal.*

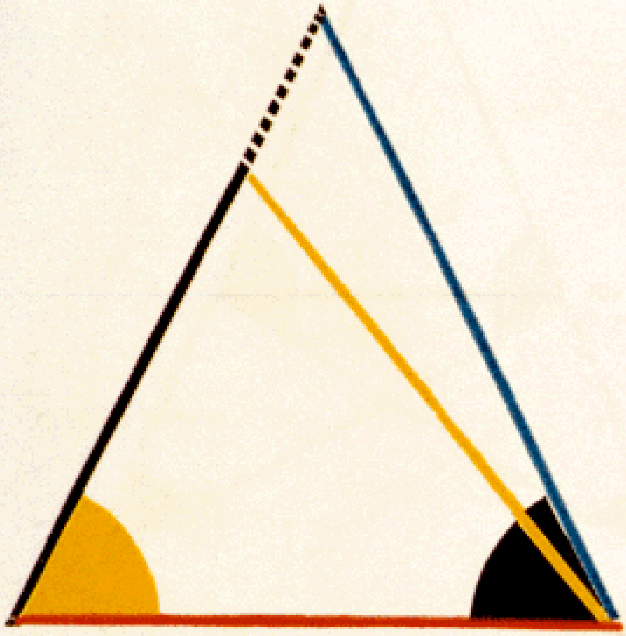
Produce , and , (post. 2.), take = , (pr. 3.); draw and .

Then in  and  we have,

 = (const.),  common to both, and = (hyp.) ∴  = , = and  =  (pr. 4.).

Again in  and  we have = ,  =  and = , ∴  =  and  =  (pr. 4.) but  = , ∴  =  (ax. 3.)

Q. E. D.



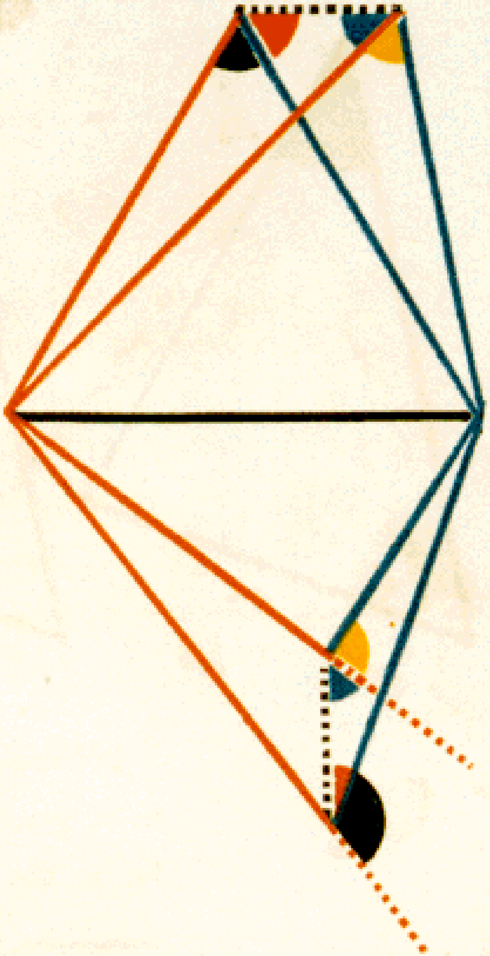
IN *any triangle ( ) if two angles (and) are equal, the ſides (and) oppoſite to them are alſo equal.*

For if the ſides be not equal, let one of them be greater than the other , and from it cut off = (pr. 3.), draw .

Then in and , = , (conſt.) = (hyp.) and common,

∴ the triangles are equal (pr. 4.) a part equal to the whole, which is abſurd; ∴ neither of the ſides or is greater than the other, ∴ hence they are equal

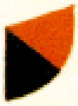
Q. E. D.

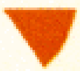
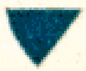


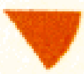
ON *the same base* (———), *and on the same side of it there cannot be two triangles having their conterminous sides* (——— *and* ———, ——— *and* ———) *at both extremities of the base, equal to each other.*

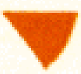
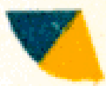
When two triangles stand on the same base, and on the same side of it, the vertex of the one shall either fall outside of the other triangle, or within it; or, lastly, on one of its sides.

If it be possible let the two triangles be constructed so that { ——— = ———, ——— = ——— }, then draw - - - - - - and,

 =  (pr. 5.)

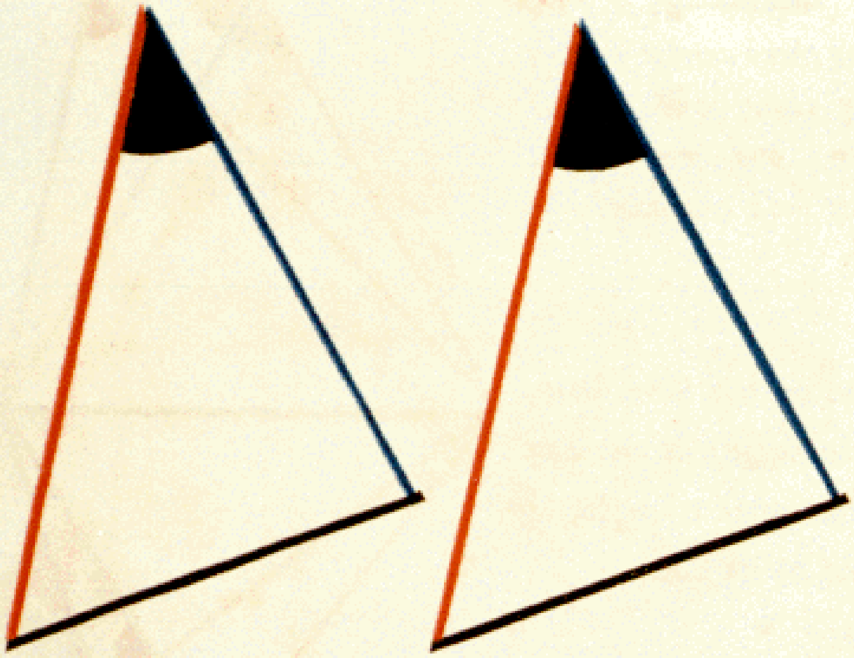
∴  ⊐  and

∴  ⊐  }

but (pr. 5.)  =  } which is absurd,

therefore the two triangles cannot have their conterminous sides equal at both extremities of the base.

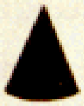
Q. E. D.



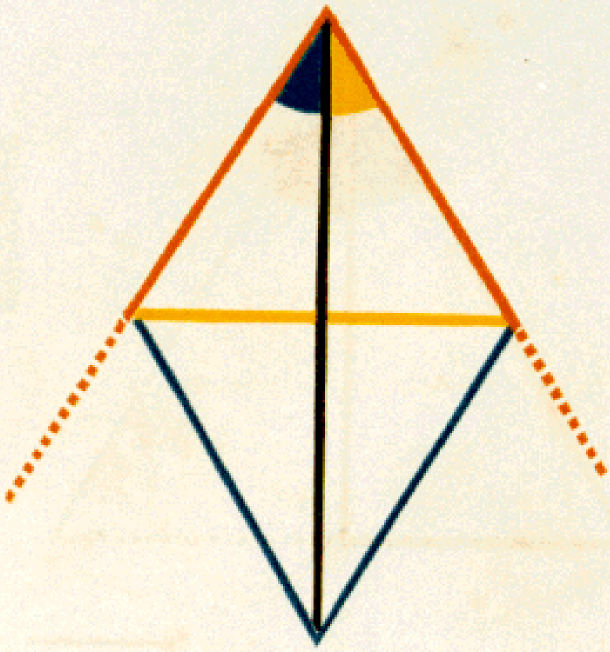
If two triangles have two ſides of the one reſpectively equal to two ſides of the other (—— = —— *and* —— = ——), *and alſo their baſes* (—— = ——), *equal; then the angles* ( *and* ) *contained by their equal ſides are alſo equal.*

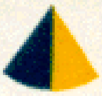
If the equal baſes —— and —— be conceived to be placed one upon the other, ſo that the triangles ſhall lie at the ſame ſide of them, and that the equal ſides —— and ——, —— and —— be conterminous, the vertex of the one muſt fall on the vertex of the other; for to ſuppoſe them not coincident would contradict the laſt propoſition.

Therefore the ſides —— and ——, being coincident with —— and ——,

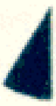
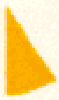
∴  = .

Q. E. D.

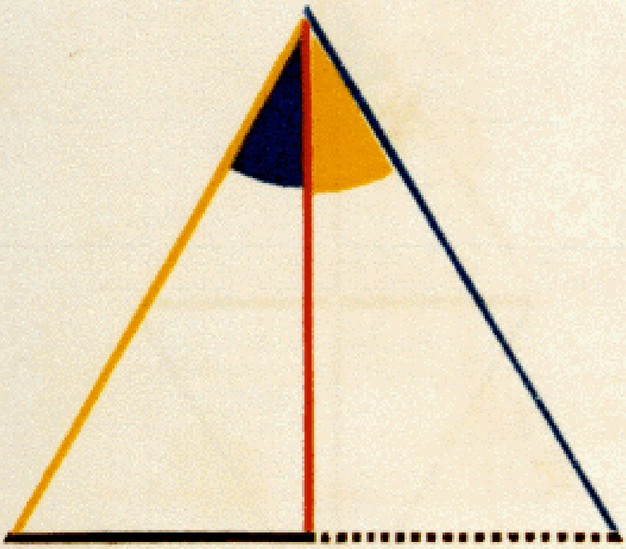


TO biſect a given rectilinear angle ().

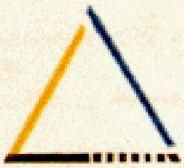
Take = (pr. 3.)
draw , upon which
deſcribe  (pr. 1.),
draw .

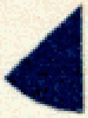
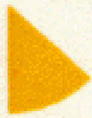
Becauſe = (conſt.)
and common to the two triangles
and =  (conſt.),
∴  =  (pr. 8.)

Q. E. D.

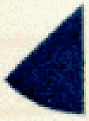
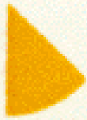


T*O bisect a given finite straight line* ().

Construct  (pr. 1.),

draw , making  =  (pr. 9.).

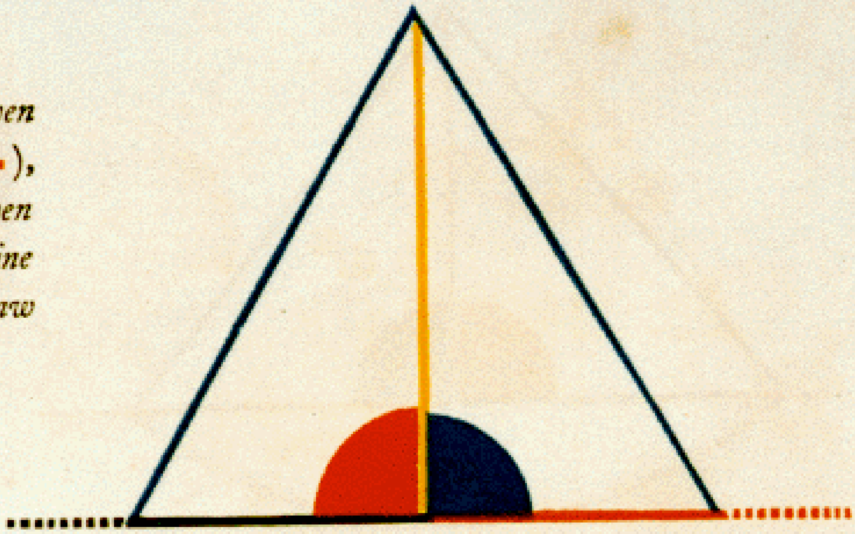
Then = by (pr. 4.),

for = (const.)  = 

and common to the two triangles.

Therefore the given line is bisected.

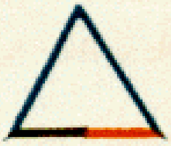
Q. E. D.



FROM *a given point* (), *in a given ſtraight line* (), *to draw a perpendicular.*

Take any point () in the given line,

cut off  =  (pr. 3.),

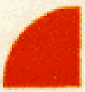
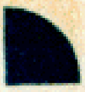
conſtruct  (pr. 1.),

draw and it ſhall be perpendicular to the given line.

For = (conſt.)

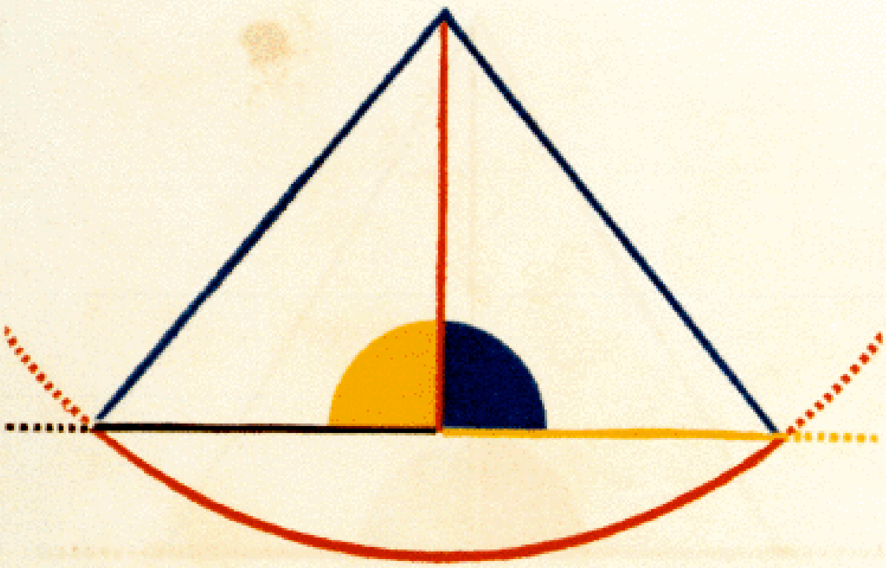
 = (conſt.)

and common to the two triangles.

Therefore  =  (pr. 8.)

∴ ⊥  (def. 10.).

Q. E. D.



TO draw a ſtraight line perpendicular to a given indefinite ſtraight line ( ) from a given (point) without.

With the given point as centre, at one ſide of the line, and any diſtance capable of extending to the other ſide, deſcribe ,

Make = (pr. 10.)

draw , and .

then ⊥ .

For (pr. 8.) ſince = (conſt.)

common to both,

and = (def. 15.)

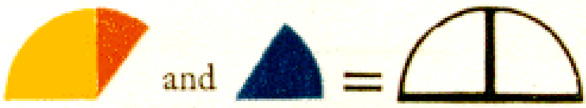
∴ = , and

∴ ⊥ (def. 10.).

Q. E. D.

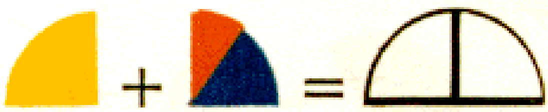
WHEN *a ſtraight line* (————) *ſtanding upon another ſtraight line* (————) *makes angles with it; they are either two right angles or together equal to two right angles.*

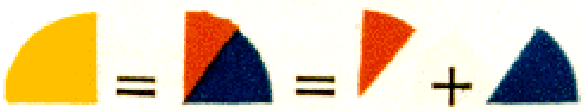
If ———— be ⊥ to ———— then,

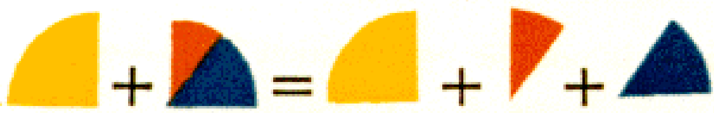
 (def. 7.),

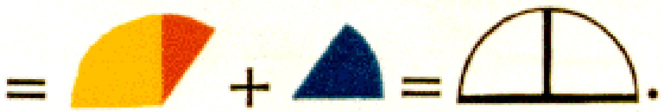
But if ———— be not ⊥ to ————,

draw ———— ⊥ ————; (pr. 11.)

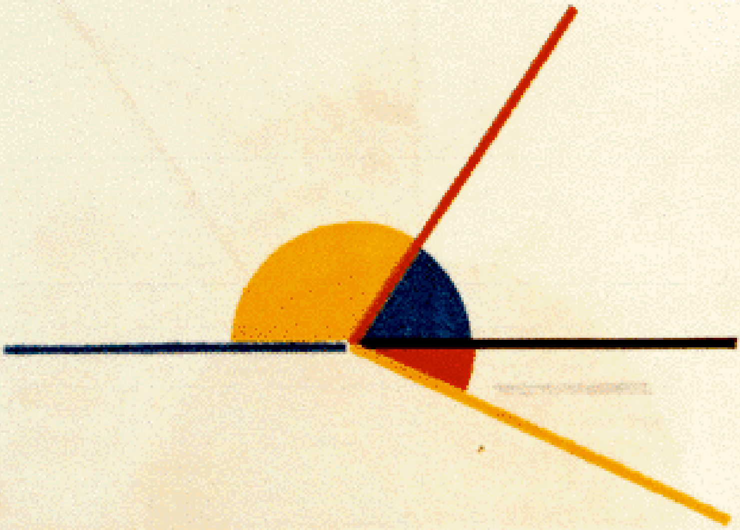
 (conſt.),



∴  (ax. 2.)

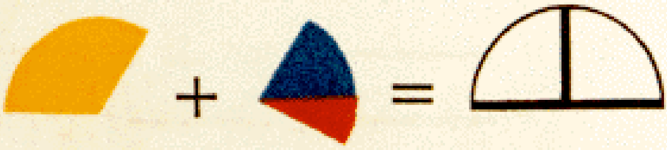


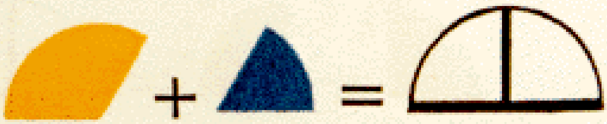
Q. E. D.

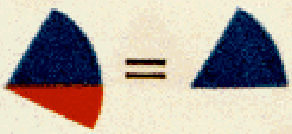


IF two ſtraight lines (and), meeting a third ſtraight line ( ), at the ſame point, and at oppoſite ſides of it, make with it adjacent angles (and) equal to two right angles; theſe ſtraight lines lie in one continuous ſtraight line.

For, if poſſible let , and not , be the continuation of ,

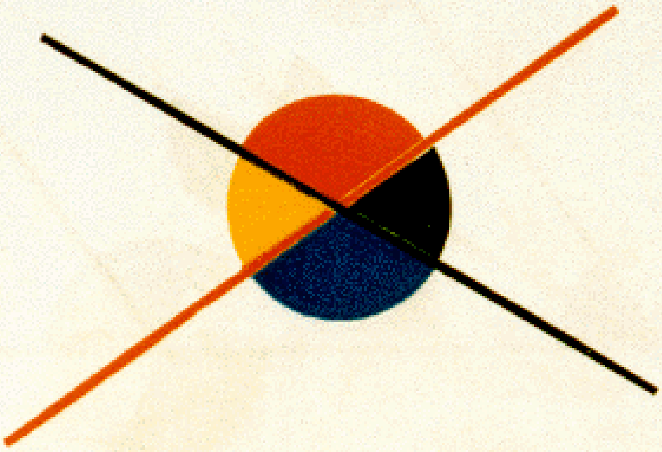
then 

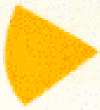
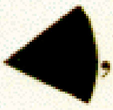
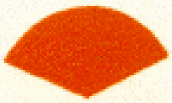
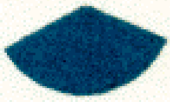
but by the hypotheſis 

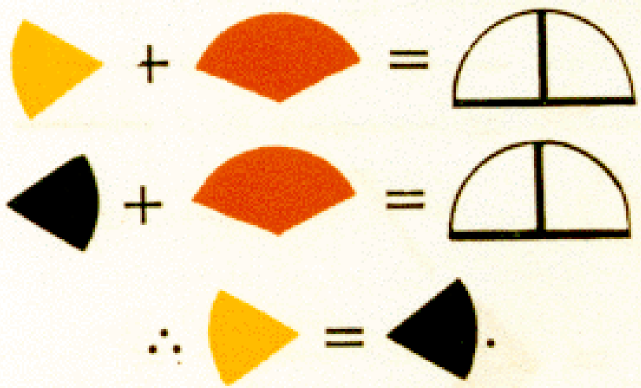
∴ , (ax. 3.); which is abſurd (ax. 9.).

∴ , is not the continuation of , and the like may be demonſtrated of any other ſtraight line except , ∴ is the continuation of .

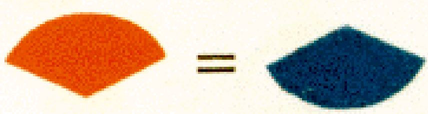
Q. E. D.



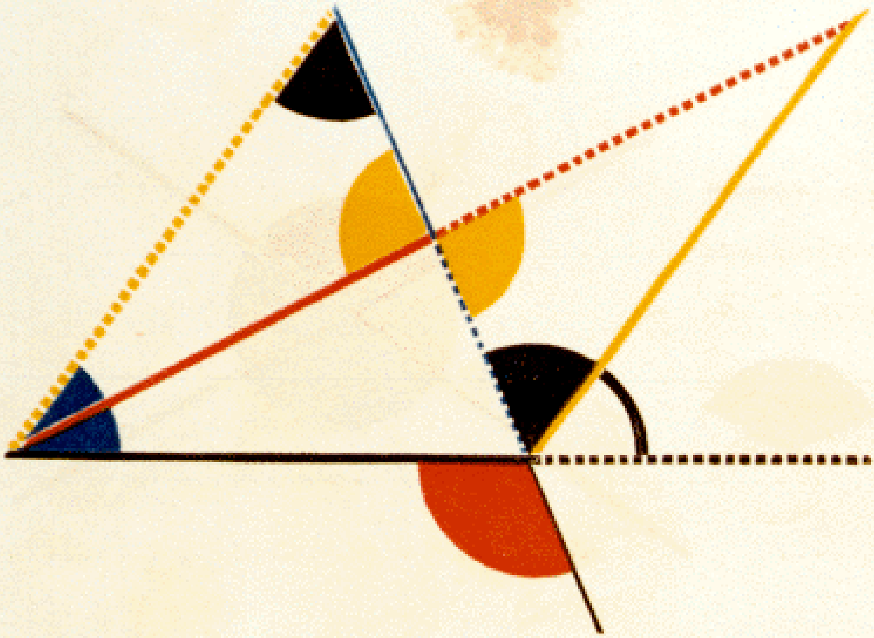
IF *two right lines* ( *and* ) *interfect one another, the vertical angles*  *and* ,  *and*  *are equal.*



In the fame manner it may be fhown that



Q. E. D.



IF a ſide of a triangle ( ) is produced, the external angle ( ) is greater than either of the internal remote angles (or).

Make = (pr. 10.).

Draw and produce it until = ; draw .

In and ; =

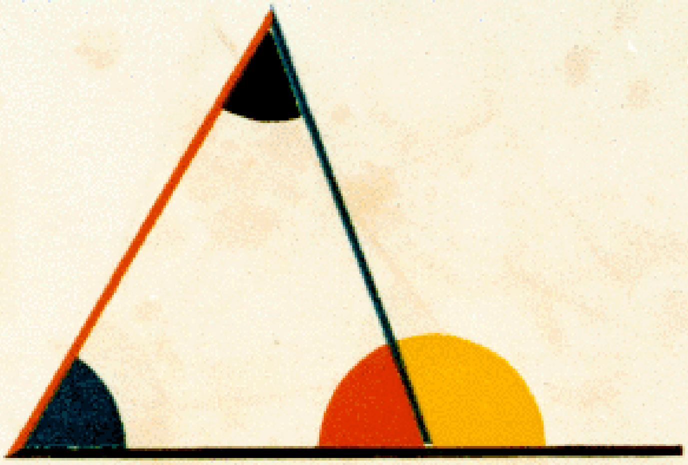
= and =

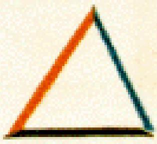
(conſt. pr. 15.), ∴ = (pr. 4.),

∴ ⊏ .

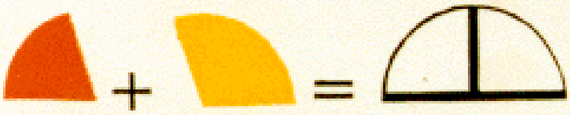
In like manner it can be ſhown, that if be produced, ⊏ , and therefore which is = is ⊏ .

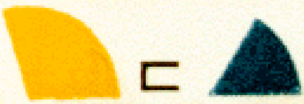
Q. E. D.

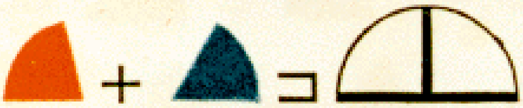


ANY *two angles of a tri-*  *angle* *are to-gether lefs than two right angles.*

Produce ———, then will

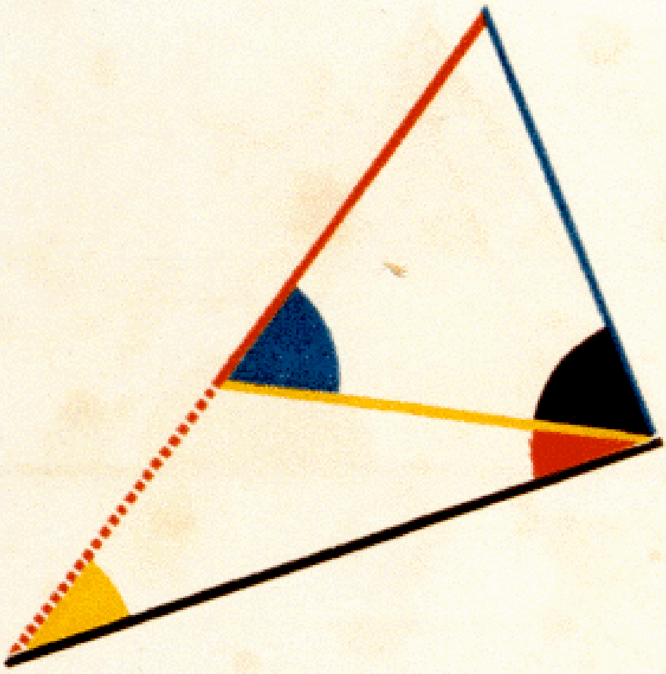


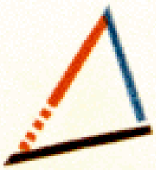
But  (pr. 16.)

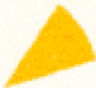
∴ ,

and in the fame manner it may be fhown that any other two angles of the triangle taken together are lefs than two right angles.

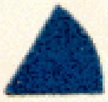
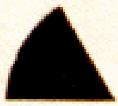
Q. E. D.

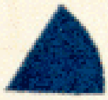
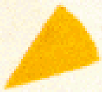


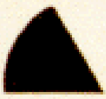
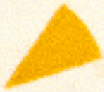
In any triangle  *if one side*  *be greater than another* , *the angle opposite to the greater side is greater than the angle opposite to the less.*

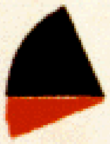
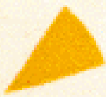
i. e.  ⊏ 

Make  = (pr. 3.), draw .

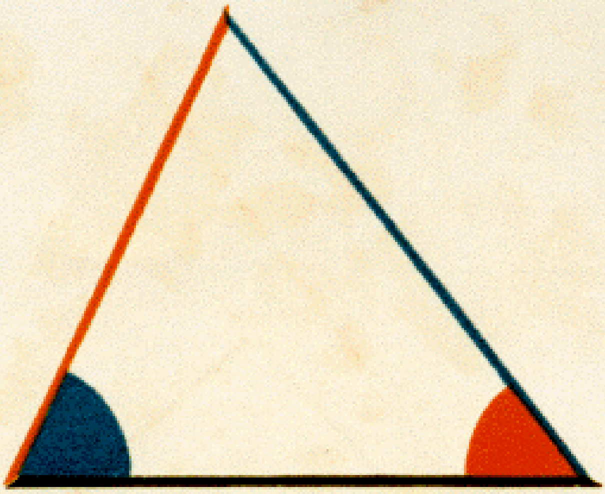
Then will  =  (pr. 5.);

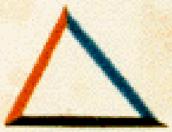
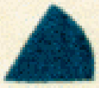
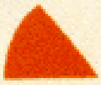
but  ⊏  (pr. 16.)

∴  ⊏  and much more

is  ⊏ .

Q. E. D.

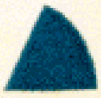
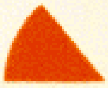


If in any triangle  *one angle*  *be greater than another*  *the side* *which is opposite to the greater angle, is greater than the side* *opposite the less.*

If be not greater than then must

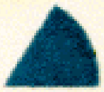
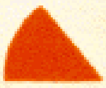
 = or ⊐ .

If = then

 =  (pr. 5.);

which is contrary to the hypothesis.

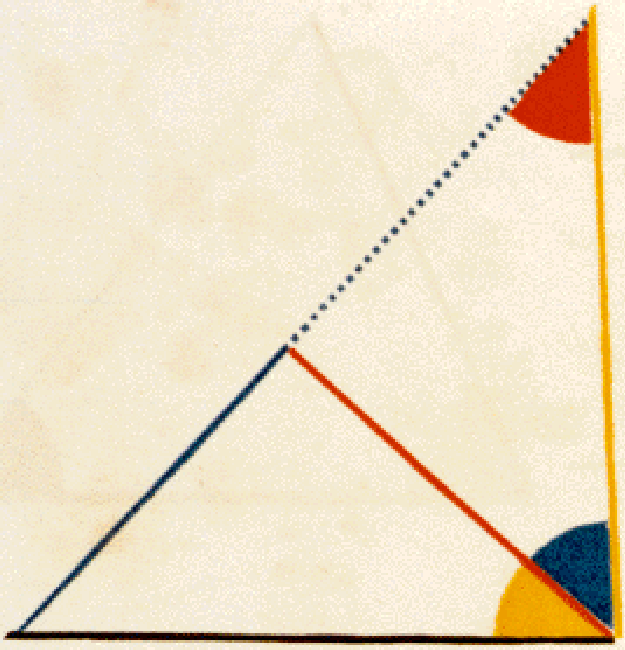
 is not less than ; for if it were,

 ⊐  (pr. 18.)

which is contrary to the hypothesis:

∴ ⊏ .

Q. E. D.



ANY *two ſides* and of a triangle taken together are greater than the third ſide ( ).

Produce , and

make = (pr. 3.);

draw .

Then becauſe = (conſt.),

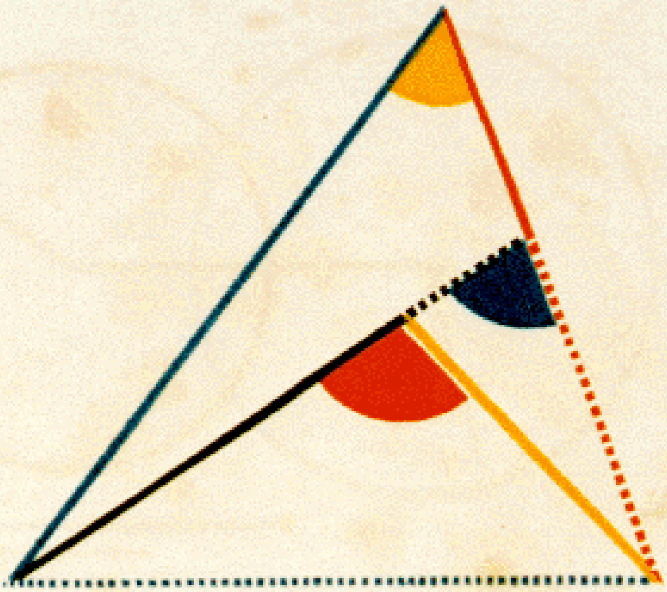
= (pr. 5.)

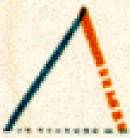
∴ ⊏ (ax. 9.)

∴ + ⊏ (pr. 19.)

and ∴ + ⊏ .

Q. E. D

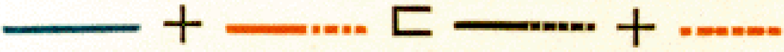


I*F from any point* () *within a triangle*  *ſtraight lines be drawn to the extremities of one ſide* (), *theſe lines muſt be together leſs than the other two ſides, but muſt contain a greater angle.*

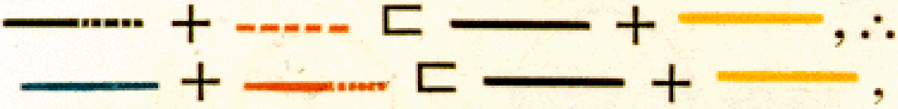
Produce ,

 (pr. 20.),

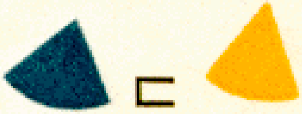
add to each,

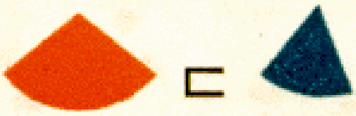
 (ax. 4.)

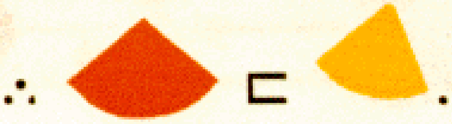
In the ſame manner it may be ſhown that



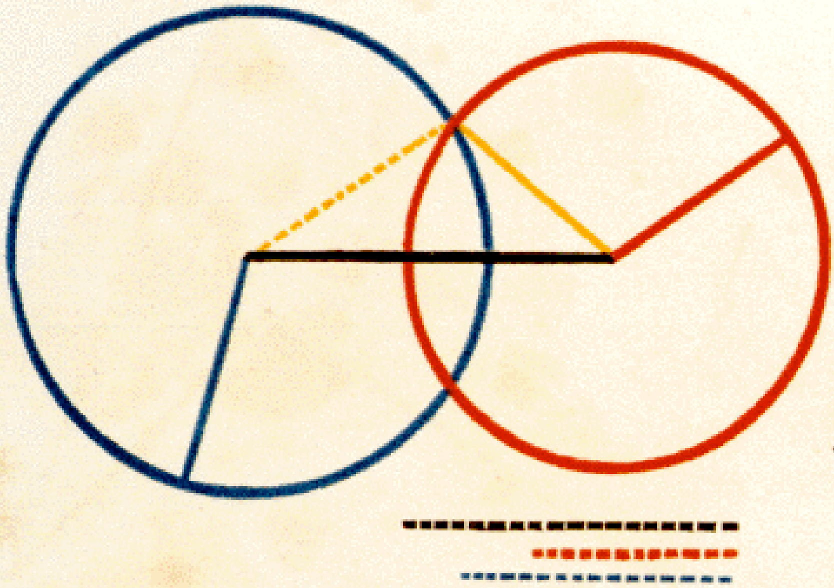
which was to be proved.

Again  (pr. 16.),

and alſo  (pr. 16.),



Q. E. D.

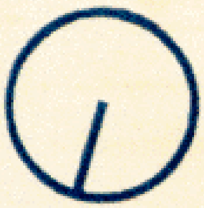


GIVEN three right lines { the ſum of any two greater than the third, to conſtruct a triangle whoſe ſides ſhall be reſpectively equal to the given lines.

Aſſume = (pr. 3.).

Draw =
and = } (pr. 2.).

With and as radii,

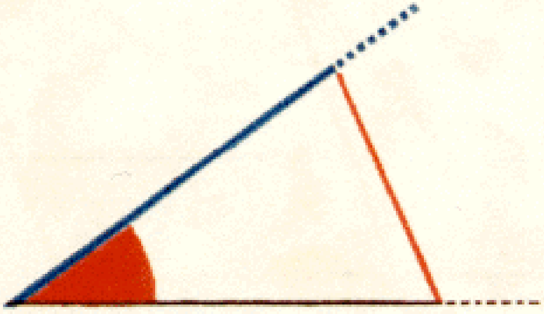
deſcribe  and  (poſt. 3.);

draw and ,

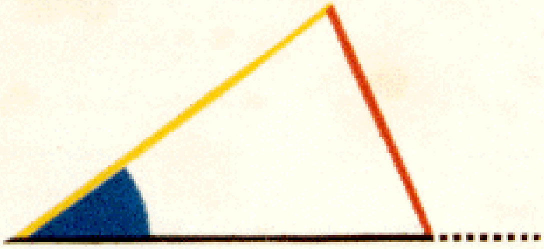
then will  be the triangle required.

For = ,
= = ,
and = = . } (conſt.)

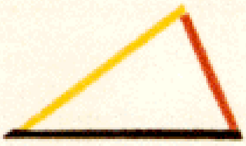
Q. E. D.



AT a given point () in a given ſtraight line (), to make an angle equal to a given rectilineal angle ().

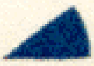
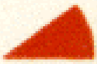


Draw between any two points in the legs of the given angle.

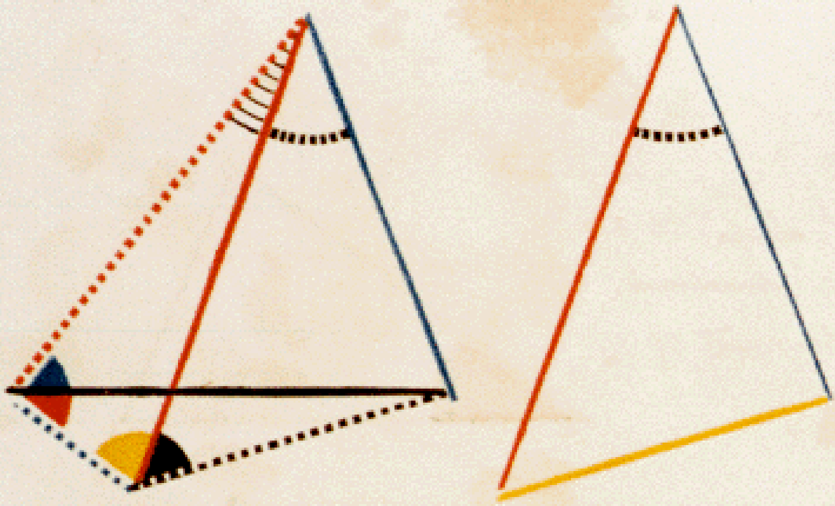
Conſtruct  (pr. 22.).

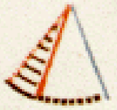
ſo that = , =

and .

Then  =  (pr. 8.).

Q. E. D.



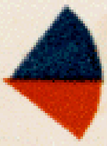
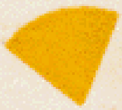
IF two triangles have two sides of the one respectively equal to two sides of the other ( *to* *and* *to* *), and if one of the angles (*  *) contained by the equal sides be greater than the other (*  *), the side (* *) which is opposite to the greater angle is greater than the side (* *) which is opposite to the less angle.*

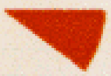
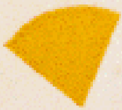
Make  =  (pr. 23.),

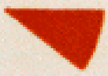
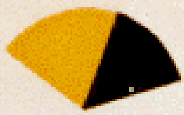
and = (pr. 3.),

draw and .

Because  = (ax. 1. hyp. const.)

∴  =  (pr. 5.)

but  ⊐ 

and ∴  ⊐ ,

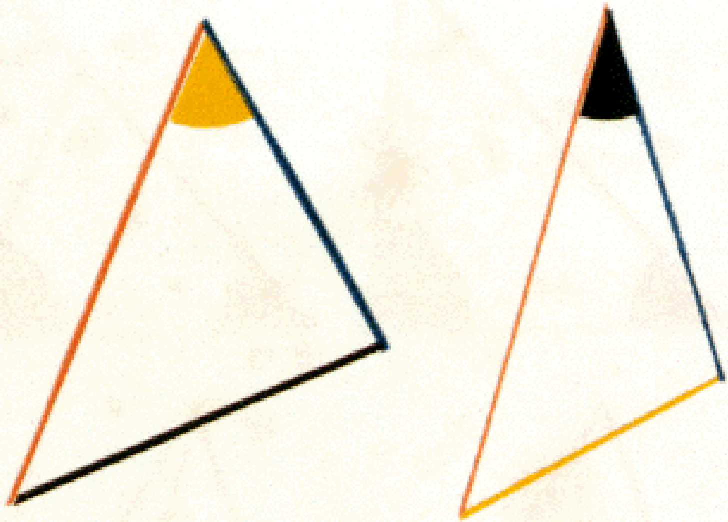
∴  ⊏  (pr. 19.)

but = (pr. 4.)

∴ ⊏ .

Q. E. D.

If two triangles have two sides (and) of the one respectively equal to two sides (and) of the other, but their bases unequal, the angle subtended by the greater base () of the one, must be greater than the angle subtended by the less base () of the other.



=, ⊏ or ⊐   is not equal to

for if = then = (pr. 4.)

which is contrary to the hypothesis;

is not less than

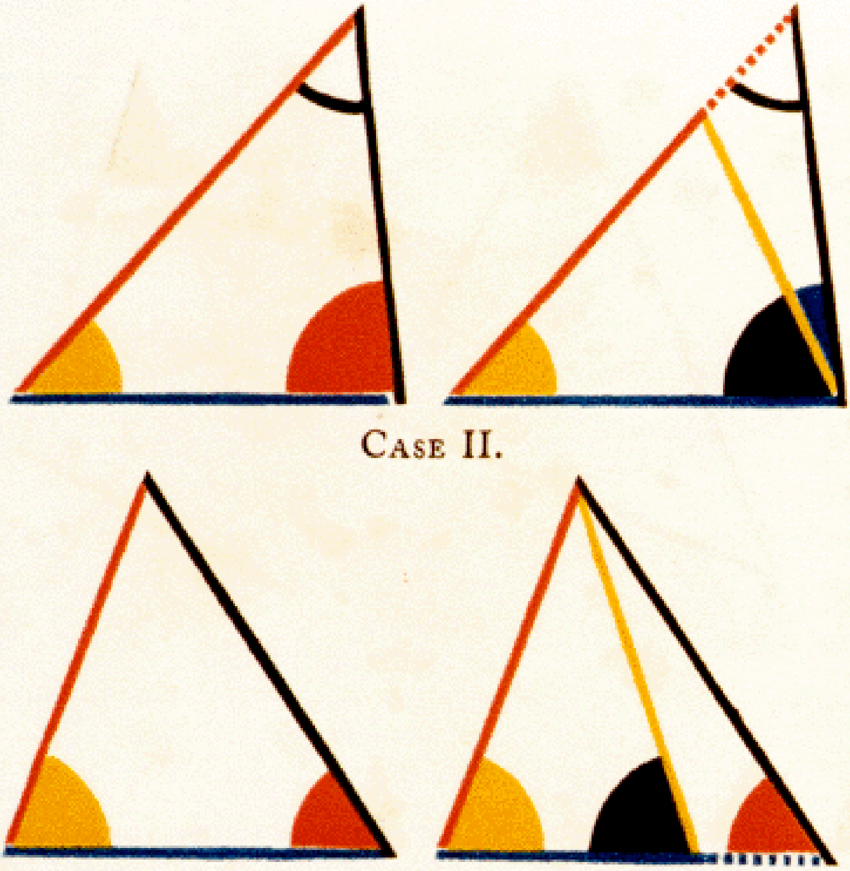
for if ⊐

then ⊐ (pr. 24.),

which is also contrary to the hypothesis:

∴ ⊏ .

Q. E. D.



If two triangles have two angles of the one respectively equal to two angles of the other, (= and =), and a side of the one equal to a side of the other similarly placed with respect to the equal angles, the remaining sides and angles are respectively equal to one another.

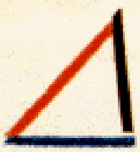
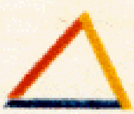
CASE I.

Let and which lie between the equal angles be equal,

then = .

For if it be possible, let one of them be greater than the other;

make = , draw .

In  and  we have

= , = , = ;

∴ = (pr. 4.)

but = (hyp.)

and therefore = , which is abſurd;
hence neither of the ſides and is
greater than the other; and ∴ they are equal;

∴ = , and = , (pr. 4.).

CASE II.

Again, let = , which lie oppoſite
the equal angles and . If it be poſſible, let
⊏ , then take = ,
draw .

Then in and we have = ,
= and = ,

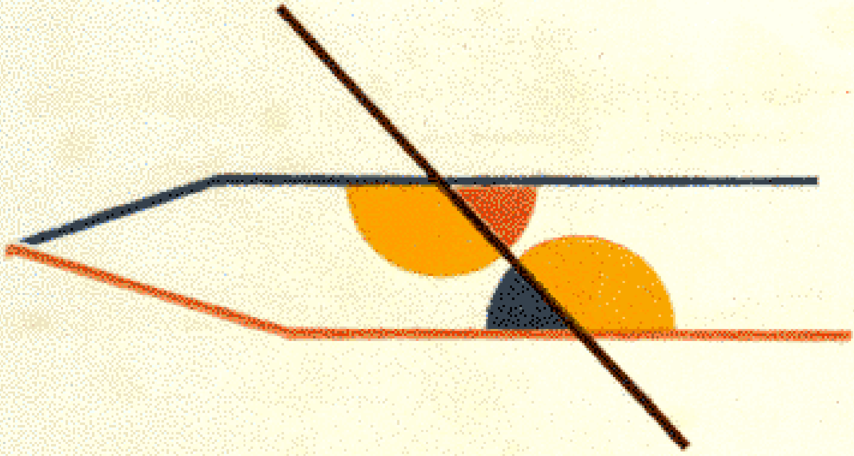
∴ = (pr. 4.)

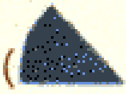
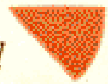
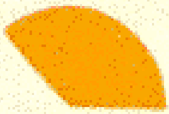
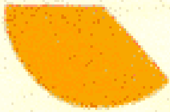
but = (hyp.)

∴ = which is abſurd (pr. 16.).

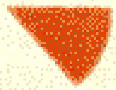
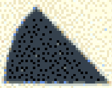
Conſequently, neither of the ſides or is greater than the other, hence they muſt be equal. It follows (by pr. 4.) that the triangles are equal in all reſpects.

Q. E. D.



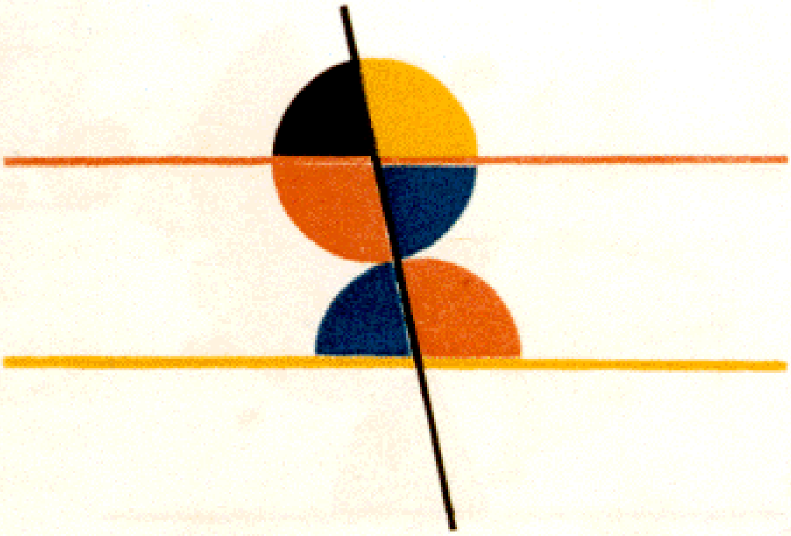
IF a ſtraight line () meeting two other ſtraight lines, ( and) makes with them the alternate angles ( and ;  and ) equal, theſe two ſtraight lines are parallel.

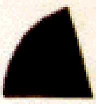
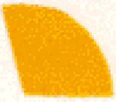
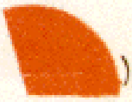
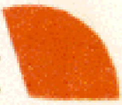
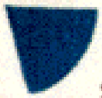
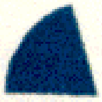
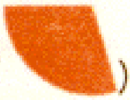
If be not parallel to they ſhall meet when produced.

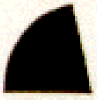
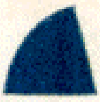
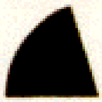
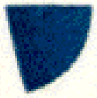
If it be poſſible, let thoſe lines be not parallel, but meet when produced; then the external angle  is greater than  (pr. 16), but they are alſo equal (hyp.), which is abſurd: in the ſame manner it may be ſhown that they cannot meet on the other ſide; ∴ they are parallel.

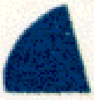
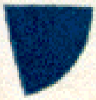
Q. E. D.

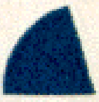
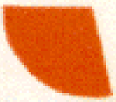
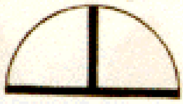
IF a ſtraight line ( ), cutting two other ſtraight lines (and), makes the external equal to the internal and oppoſite angle, at the ſame ſide of the cutting line (namely,

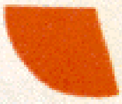
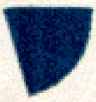
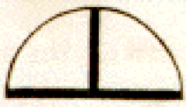


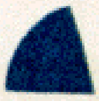
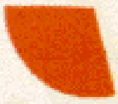
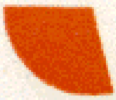
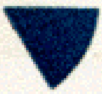
 =  *or*  = *), or if it makes the two internal angles at the ſame ſide (*  *and* *, or*  *and* *) together equal to two right angles, thoſe two ſtraight lines are parallel.*

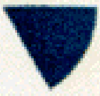
Firſt, if  = , then  =  (pr. 15.),

∴  =  ∴ || (pr. 27.).

Secondly, if  +  = ,

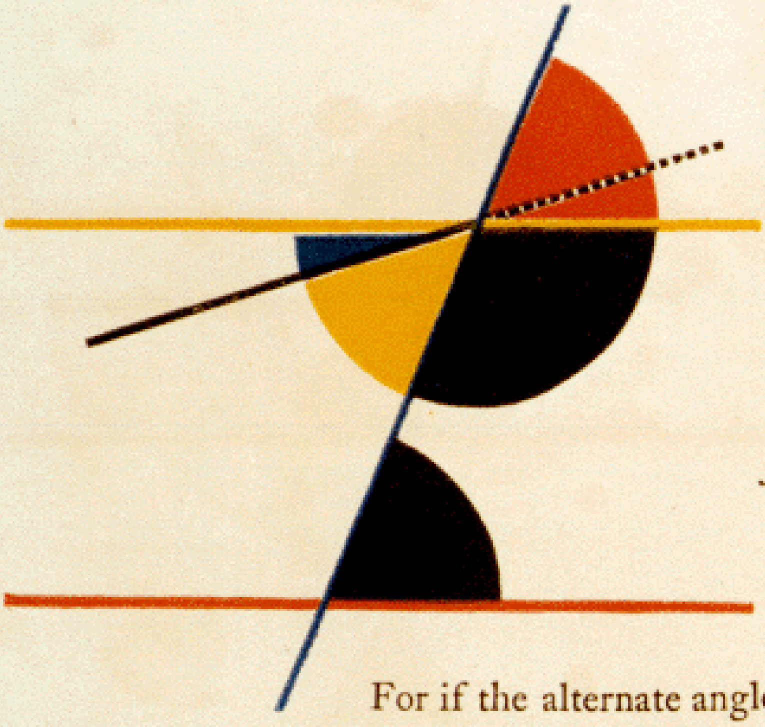
then  +  =  (pr. 13.),

∴  +  =  +  (ax. 3.)

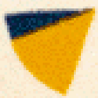
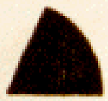
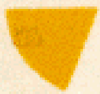
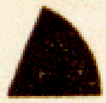
∴  = 

∴ || (pr. 27.)

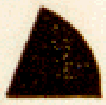
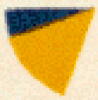
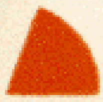
Q. E. D.

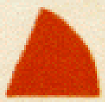
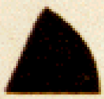
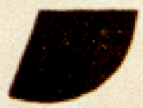
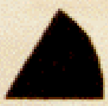
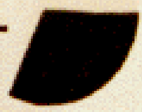
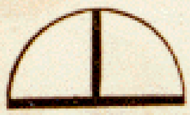


A STRAIGHT *line* ( ) *falling on two parallel ſtraight lines* (*and*), *makes the alternate angles equal to one another; and alſo the external equal to the internal and oppoſite angle on the ſame ſide; and the two internal angles on the ſame ſide together equal to two right angles.*

For if the alternate angles  and  be not equal, draw , making  =  (pr. 23).

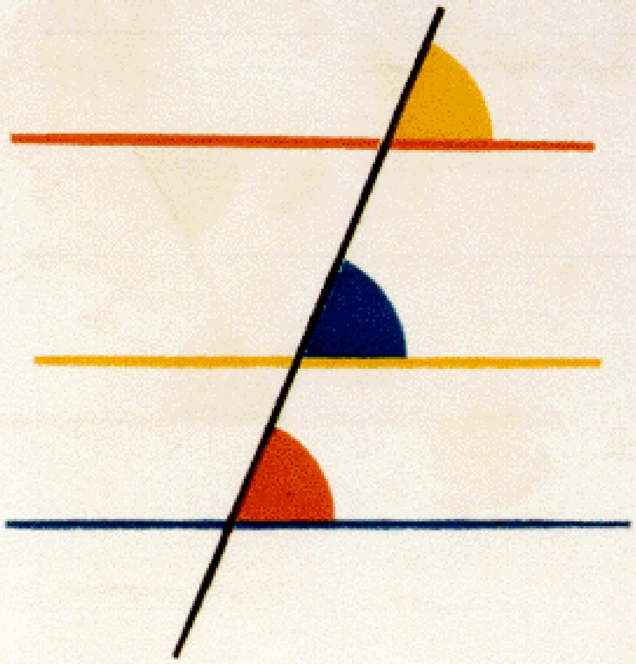
Therefore || (pr. 27.) and therefore two ſtraight lines which interſect are parallel to the ſame ſtraight line, which is impoſſible (ax. 12).

Hence the alternate angles  and  are not unequal, that is, they are equal:  =  (pr. 15);

∴  = , the external angle equal to the internal and oppoſite on the ſame ſide: if  be added to both, then  +  =  =  (pr. 13).

That is to ſay, the two internal angles at the ſame ſide of the cutting line are equal to two right angles.

Q. E. D.



STRAIGHT *lines* ( ) *which are parallel to the ſame ſtraight line* ( ), *are parallel to one another.*

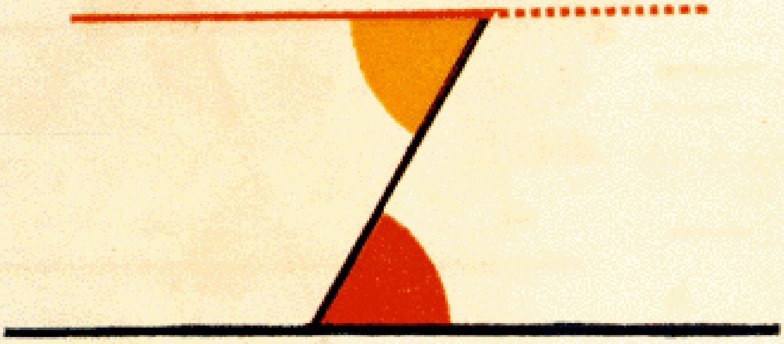
Let interſect { };

Then, = = (pr. 29.),

∴ =

∴ || (pr. 27.)

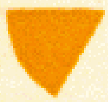
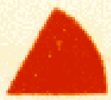
Q. E. D.



FROM *a given point*  *to draw a ſtraight line parallel to a given ſtraight line* ().

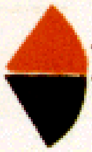
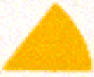
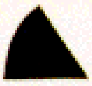
Draw from the point  to any point 

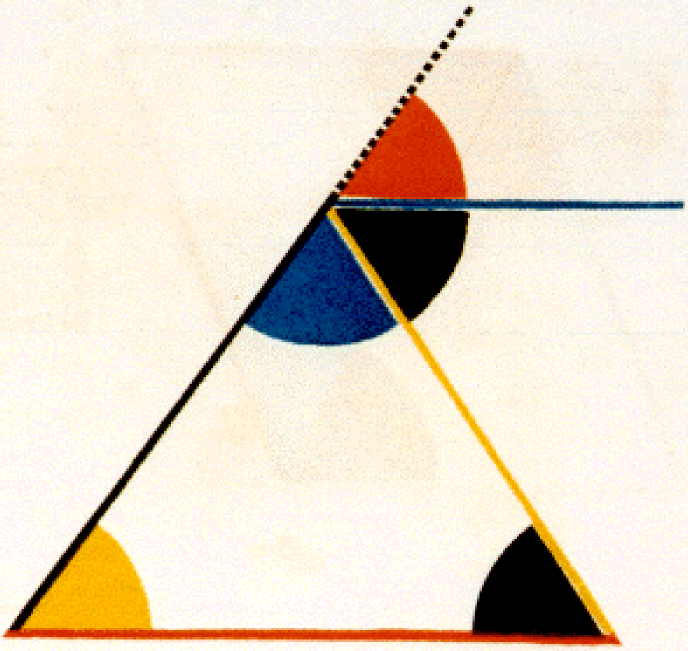
in ,

make  =  (pr. 23.),

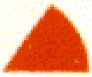
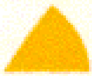
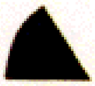
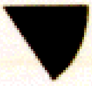
then ||  (pr. 27.).

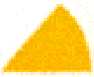
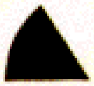
Q. E. D.

IF any side () of a triangle be produced, the external angle () is equal to the sum of the two internal and opposite angles ( and ), and the three internal angles of every triangle taken together are equal to two right angles.

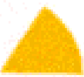
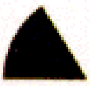
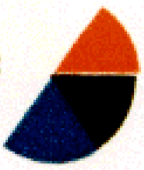
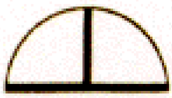


Through the point  draw || (pr. 31.).

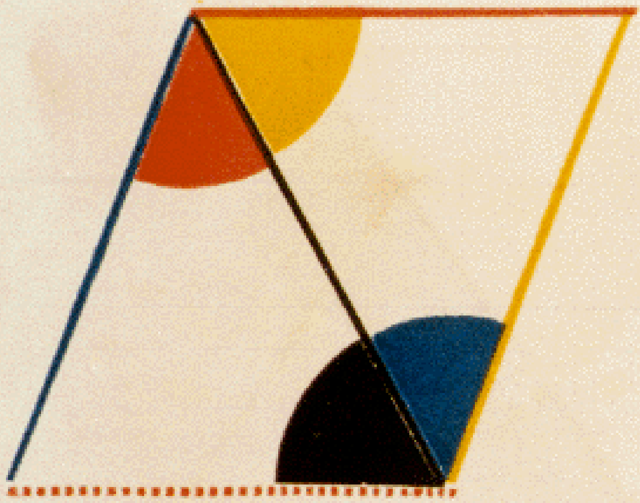
Then {  =  ,  =  } (pr. 29.),

∴  +  =  (ax. 2.),

and therefore

 +  +  =  =  (pr. 13.).

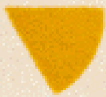
Q. E. D.



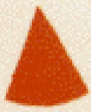
STRAIGHT *lines* (*and*) *which join the adjacent extremities of two equal and parallel ſtraight lines* (*and*), *are themſelves equal and parallel.*

Draw the diagonal.

= (hyp.)

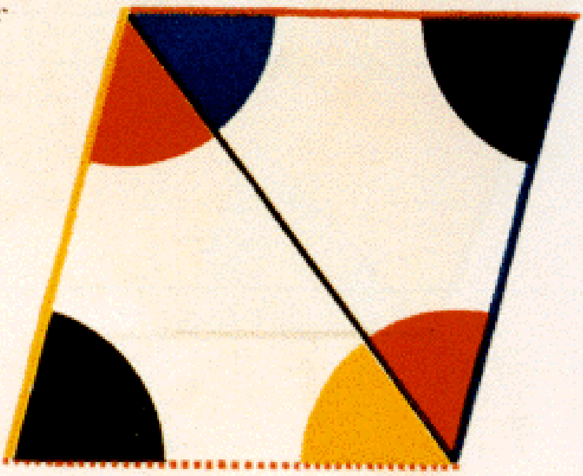
 =  (pr. 29.)

and common to the two triangles;

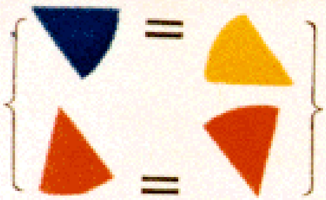
∴ = , and  =  (pr. 4.);

and ∴ || (pr. 27.).

Q. E. D.

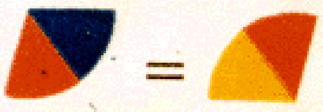


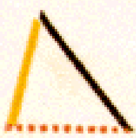
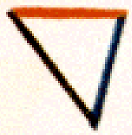
THE *oppoſite ſides and angles of any parallelogram are equal, and the diagonal () divides it into two equal parts.*

Since  (pr. 29.)

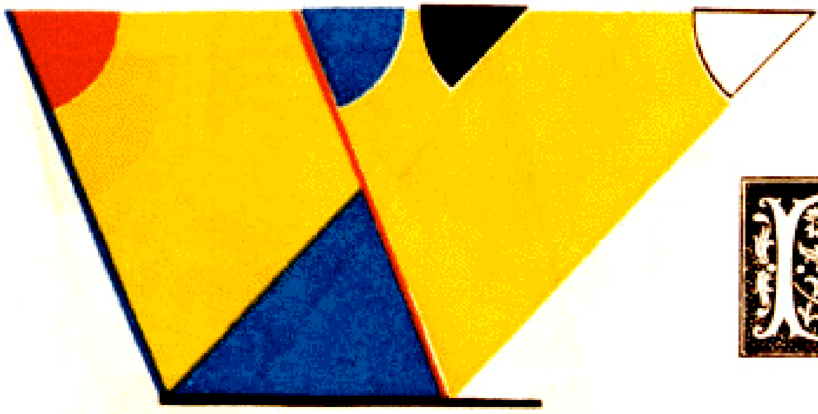
and common to the two triangles.

∴  (pr. 26.)

and  (ax.):

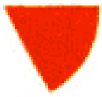
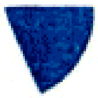
Therefore the oppoſite ſides and angles of the parallelogram are equal: and as the triangles  and  are equal in every reſpect (pr. 4.), the diagonal divides the parallelogram into two equal parts.

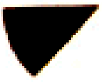
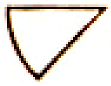
Q. E. D.



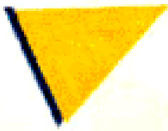
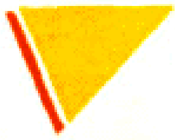
PARALLELOGRAMS *on the same base, and between the same parallels, are (in area) equal.*

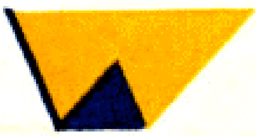
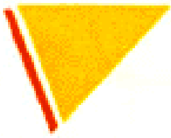
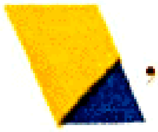
On account of the parallels,

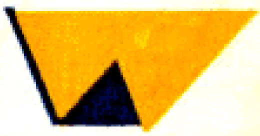
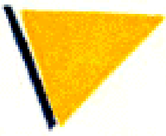
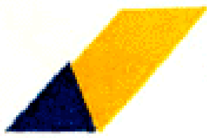
 = ; (pr. 29.)

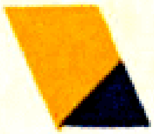
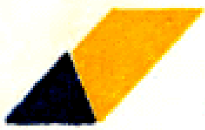
 = ; (pr. 29.)

and  = (pr. 34.)

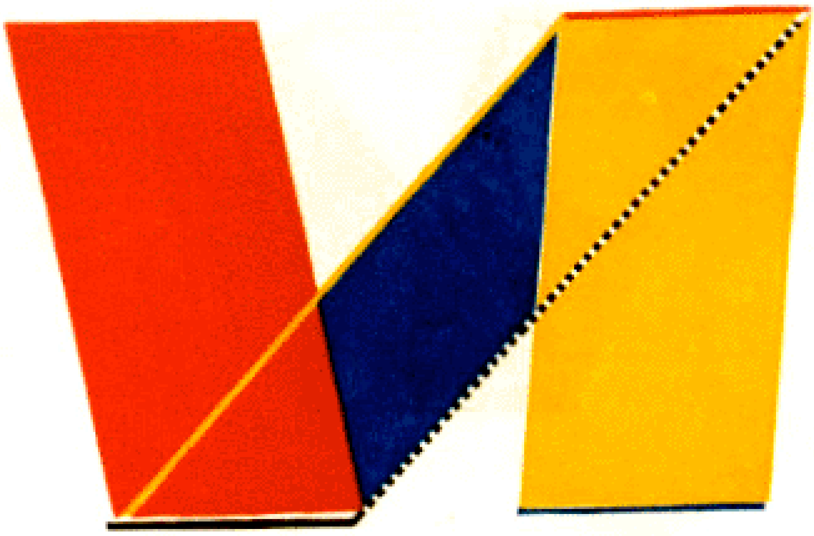
But,  =  (pr. 8.)

∴  minus  = ,

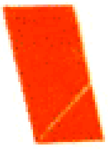
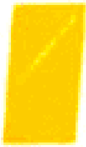
and  minus  = ;

∴  = .

Q. E. D.



PARALLELOGRAMS

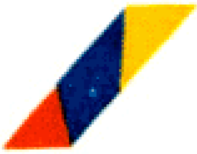
( *and* ) *on equal bafes, and between the fame parallels, are equal.*

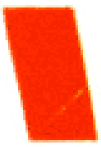
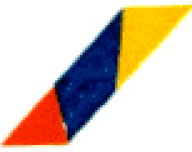
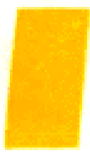
Draw and ,

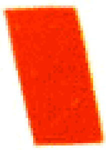
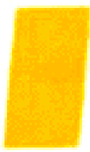
 = = , by (pr. 34, and hyp.);

∴ = and || ;

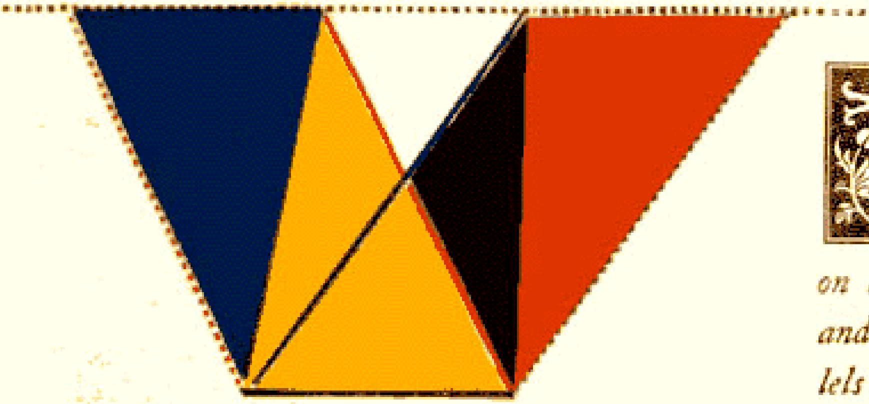
∴ = and || (pr. 33.)

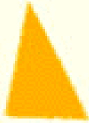
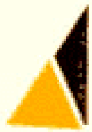
And therefore  is a parallelogram:

but  =  =  (pr. 35.)

∴  =  (ax. 1.).

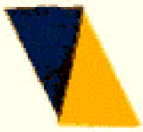
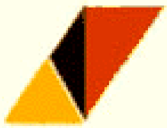
Q. E. D.

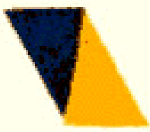
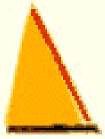
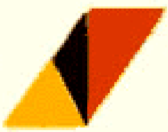
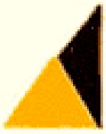


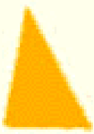
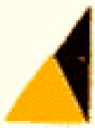
TRIANGLES  *and*  *on the same base* ( ) *and between the same parallels are equal.*

Draw ... || ——— } (pr. 31.)
... || ———

Produce ---------------.

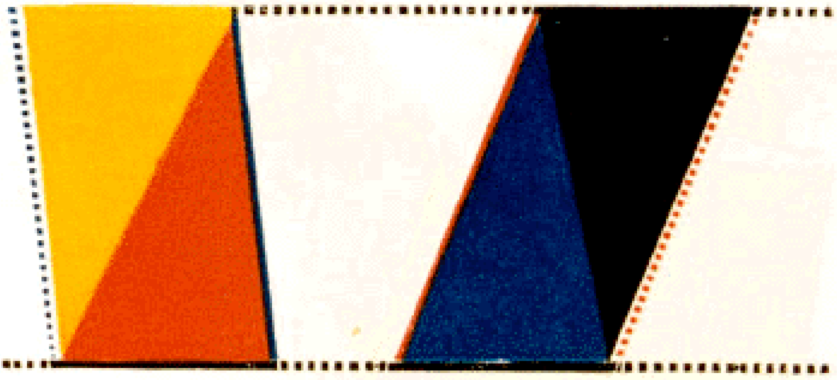
 and  are parallelograms on the same base, and between the same parallels, and therefore equal. (pr. 35.)

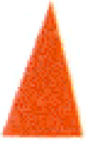
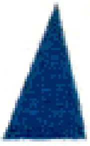
∴ {  = twice  ;  = twice  } (pr. 34.)

∴  = .

Q. E. D.

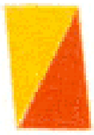
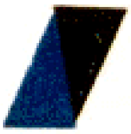
TRIANGLES

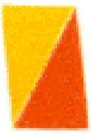
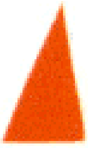


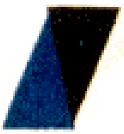
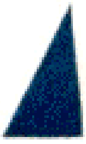
( *and* ) *on equal bases and between the same parallels are equal.*

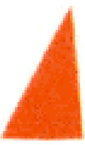
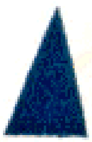
Draw ||

and || } (pr. 31.).

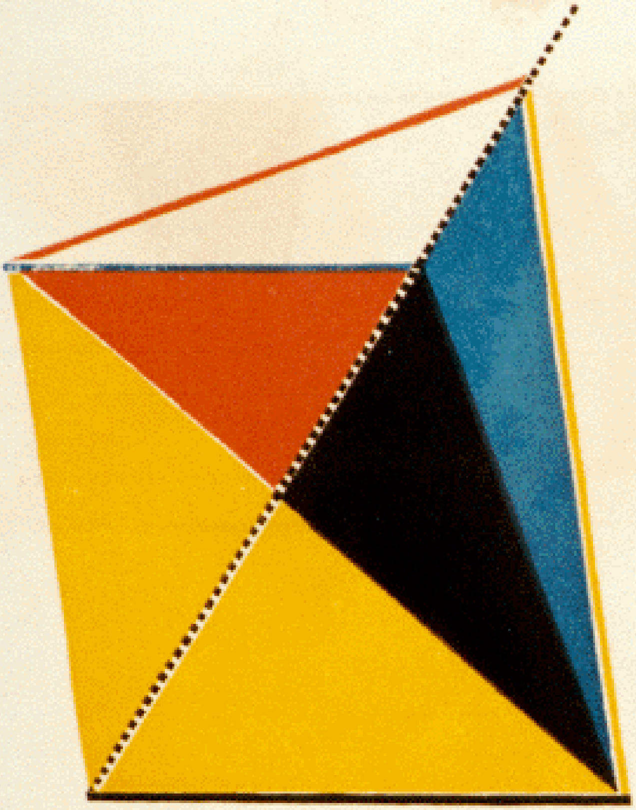
 =  (pr. 36.);

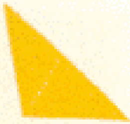
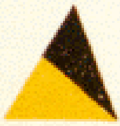
but  = twice  (pr. 34.),

and  = twice  (pr. 34.),

∴  =  (ax. 7.).

Q. E. D.

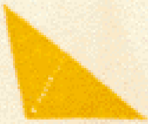
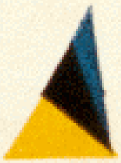


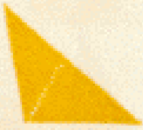
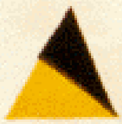
EQUAL triangles  and  *on the ſame baſe* (———) *and on the ſame ſide of it, are between the ſame parallels.*

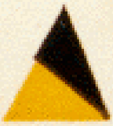
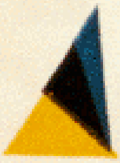
If ———, which joins the vertices of the triangles, be not || ———, draw ——— || ——— (pr. 31.), meeting - - - - - - - .

Draw ———.

Becauſe ——— || ——— (conſt.)

 =  (pr. 37.) :

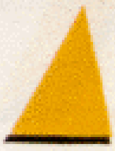
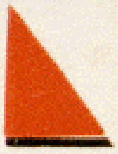
but  =  (hyp.) ;

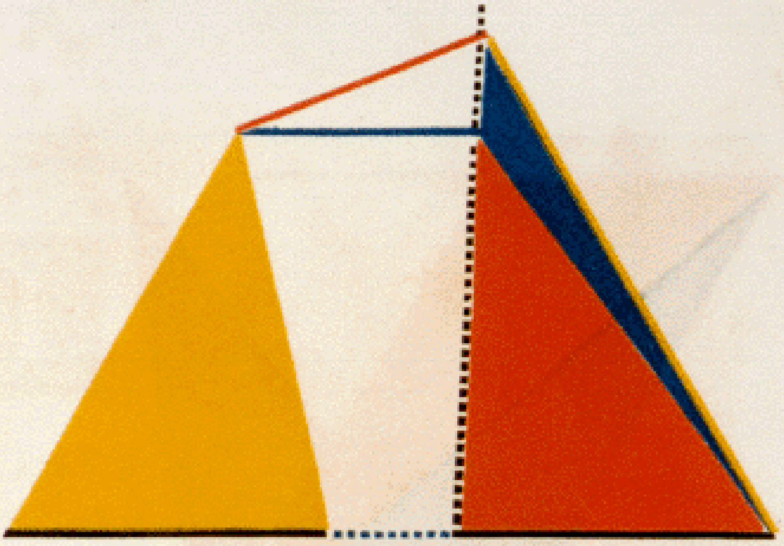
∴  = , a part equal to the whole, which is abſurd.

∴ ——— ‡ ———; and in the ſame manner it can be demonſtrated, that no other line except ——— is || ———; ∴ ——— || ———.

Q. E. D.

EQUAL *triangles*

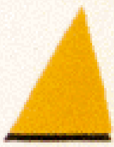
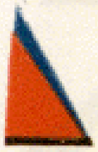
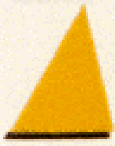
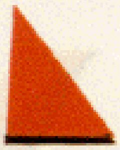
( *and* ) *on equal bases, and on the same side, are between the same parallels.*

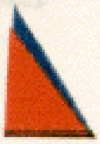


If ——— which joins the vertices of triangles be not || ———, draw ——— || ——— (pr. 31.), meeting ———.

Draw ———.

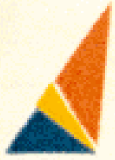
Because ——— || ——— (const.)

 =  but  = 

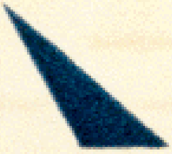
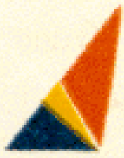
∴  = , a part equal to the whole, which is absurd.

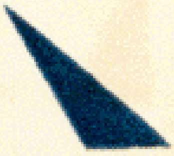
∴ ——— ∦ ———: and in the same manner it can be demonstrated, that no other line except ——— is || ———: ∴ ——— || ———.

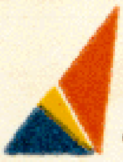
Q. E. D.

If a parallelogram  *and a triangle*  *are upon the same base and between the same parallels , the parallelogram is double the triangle.*

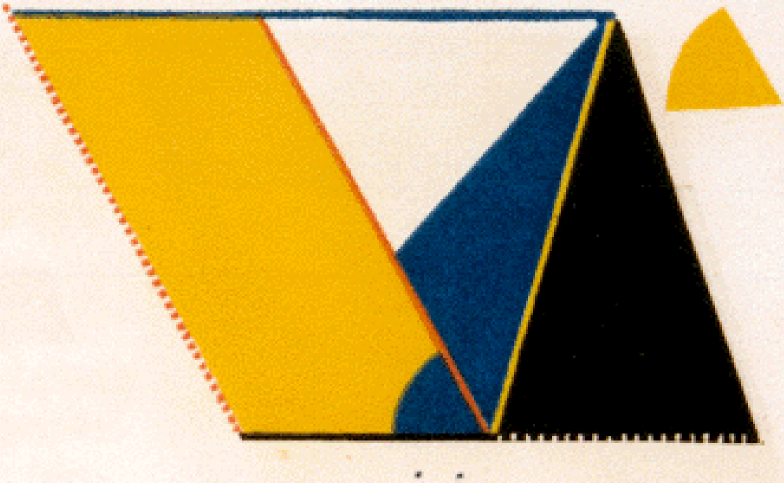
Draw the diagonal ;

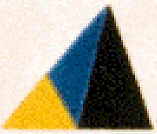
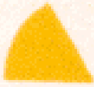
Then  =  (pr. 37.)

 = twice  (pr. 34.)

∴  = twice  .

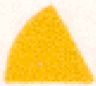
Q. E. D.

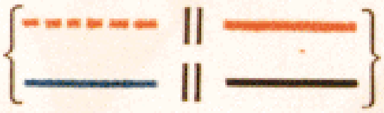


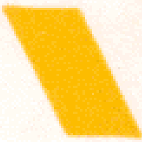
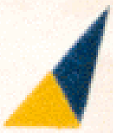
TO construct a parallelogram equal to a given triangle  *and having an angle equal to a given rectilinear angle* .

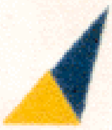
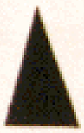
Make = (pr. 10.)

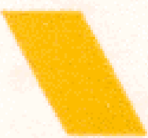
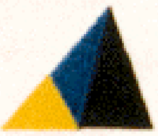
Draw .

Make  =  (pr. 23.)

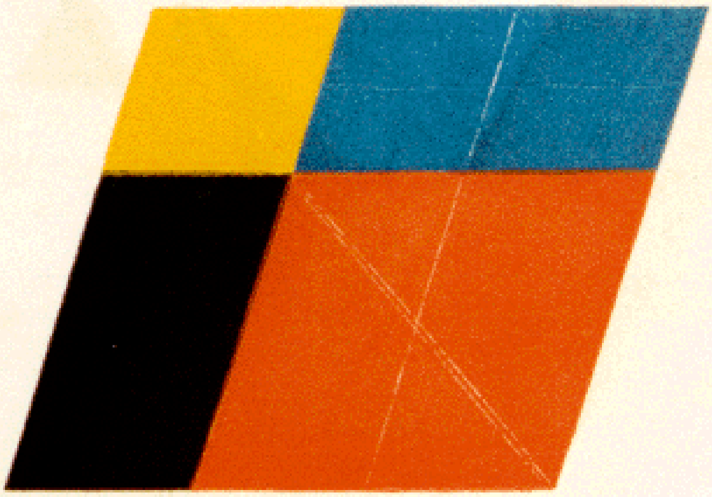
Draw  (pr. 31.)

 = twice  (pr. 41.)

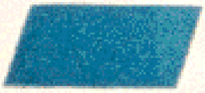
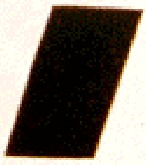
but  =  (pr. 38.)

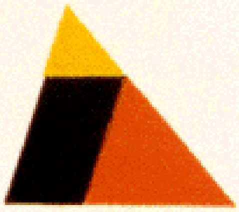
∴  = .

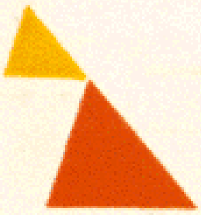
Q. E. D.

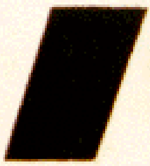
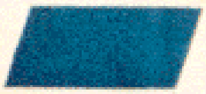


THE *complements*

 *and*  *of the parallelograms which are about the diagonal of a parallelogram are equal.*

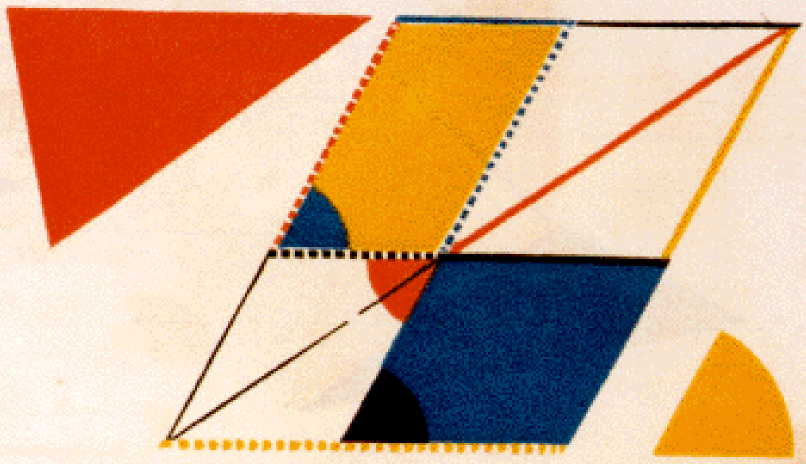
 =  (pr. 34.)

and  =  (pr. 34.)

∴  =  (ax. 3.)

Q. E. D.

TO a given ſtraight line ( ) to apply a parallelogram equal to a given triangle ( ), and having an angle equal to a given rectilinear angle ( ).



Make = with =
(pr. 42.)
and having one of its ſides conterminous
with and in continuation of .
Produce till it meets ||
draw produce it till it meets continued;
draw || meeting
produced, and produce .

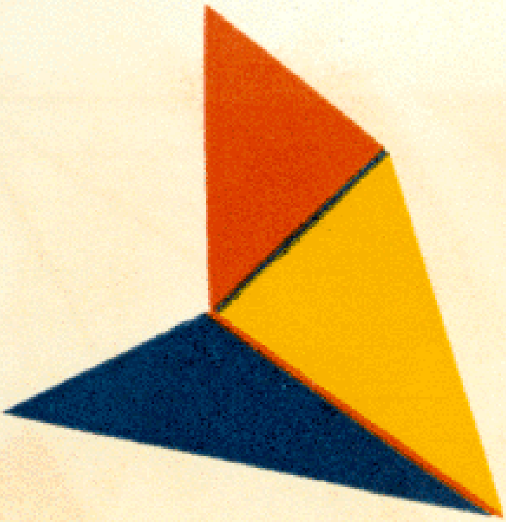
= (pr. 43.)

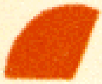
but = (conſt.)

∴ = ; and

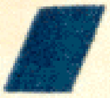
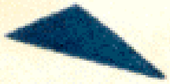
= = = (pr. 19. and conſt.)

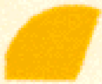
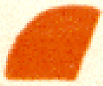
Q. E. D.

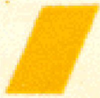
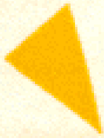


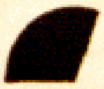
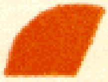
TO construct a parallelogram equal to a given rectilinear figure () *and having an angle equal to a given rectilinear angle* ().

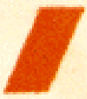
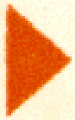
Draw  and dividing the rectilinear figure into triangles.

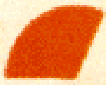
Construct  = 

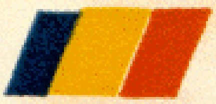
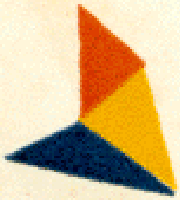
having  =  (pr. 42.)

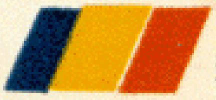
to  apply  = 

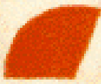
having  =  (pr. 44.)

to  apply  = 

having  =  (pr. 44.)

∴  = 

and  is a parallelogram. (prs. 29, 14, 30.)

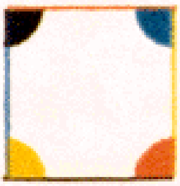
having  = .

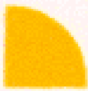
Q. E. D.

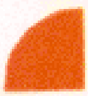
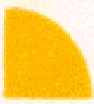
UPON *a given ſtraight line* () *to conſtruct a ſquare.*

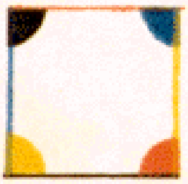
Draw ⊥ and = (pr. 11. and 3.)

Draw || , and meeting drawn || .

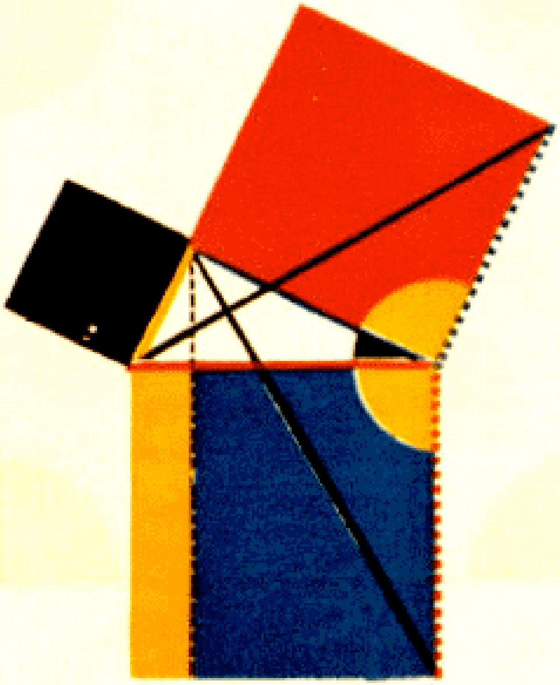
In  = (conſt.)

 = a right angle (conſt.)

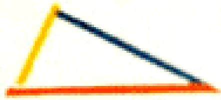
∴  =  = a right angle (pr. 29.), and the remaining ſides and angles muſt be equal, (pr. 34.)

and ∴  is a ſquare. (def. 27.)

Q. E. D.



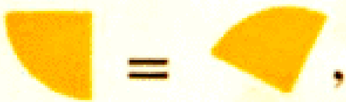
IN a right angled triangle



the square on the hypotenuse —— *is equal to the sum of the squares of the sides*, (—— *and* ——).

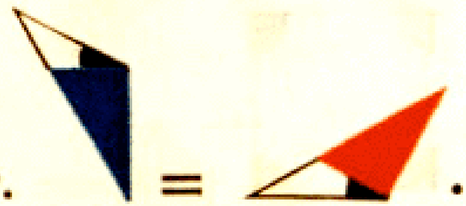
On ——, —— and —— describe squares, (pr. 46.)

Draw ······· || - - - - - (pr. 31.)
also draw —— and ——.

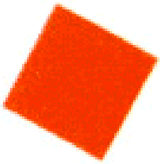
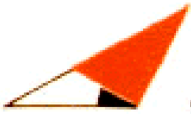


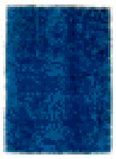
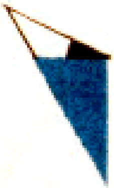
To each add  ∴ ,

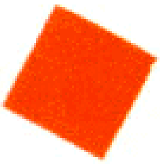
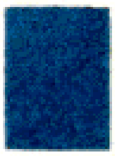
—— = - - - - - and —— = ·······;

∴ .

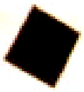
Again, because —— || ·······

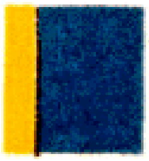
 = twice ,

and  = twice ;

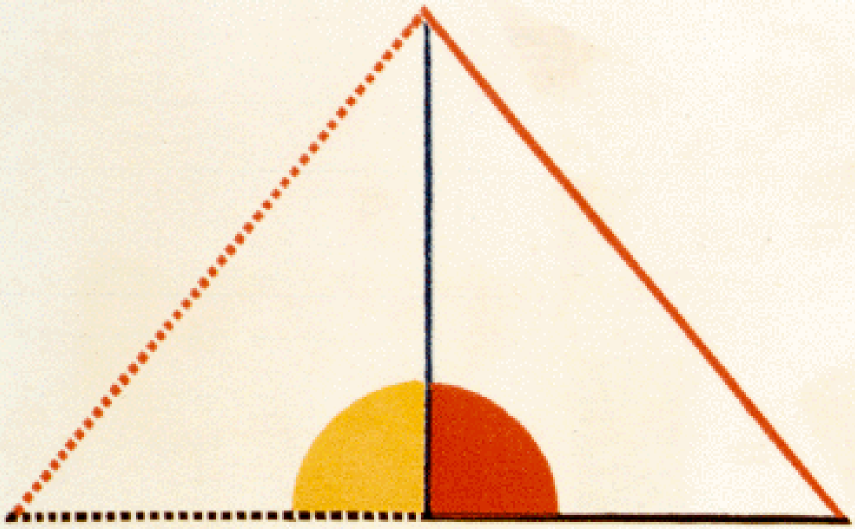
∴  = .

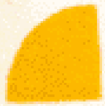
In the ſame manner it may be ſhown

that  = ;

hence  = .

Q. E. D.



IF the ſquare of one ſide (——) of a triangle is equal to the ſquares of the other two ſides (—— and ——), the angle () ſubtended by that ſide is a right angle.

Draw - - - ⊥ —— and = —— (prs. 11.3.) and draw - - - alſo.

Since - - - = —— (conſt.)

$(- - -)^2 = (——)^2$;

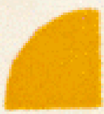
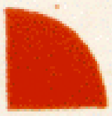
∴ $(- - -)^2 + (——)^2 = (——)^2 + (——)^2$,

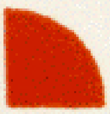
but $(- - -)^2 + (——)^2 = (- - -)^2$ (pr. 47.),

and $(——)^2 + (——)^2 = (——)^2$ (hyp.)

∴ $(- - -)^2 = (——)^2$,

∴ - - - = ——;

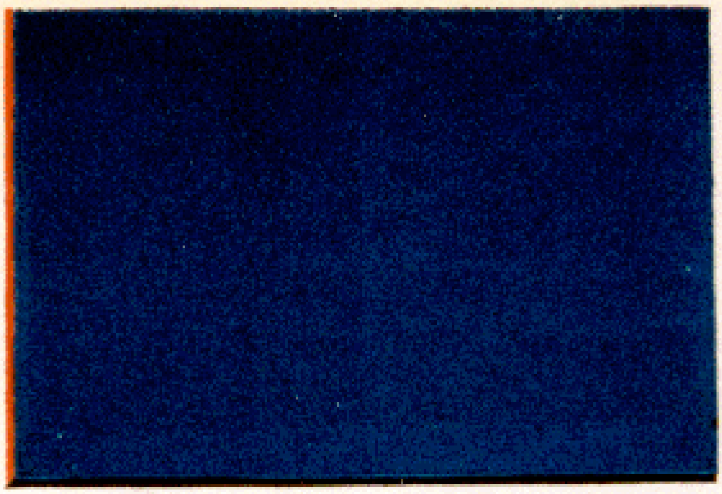
and ∴  =  (pr. 8.),

conſequently  is a right angle.

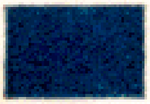
Q. E. D.

BOOK II.

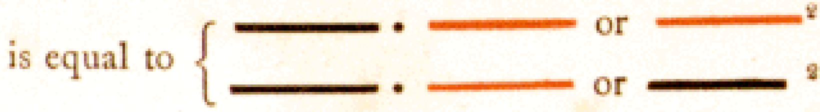
DEFINITION I.



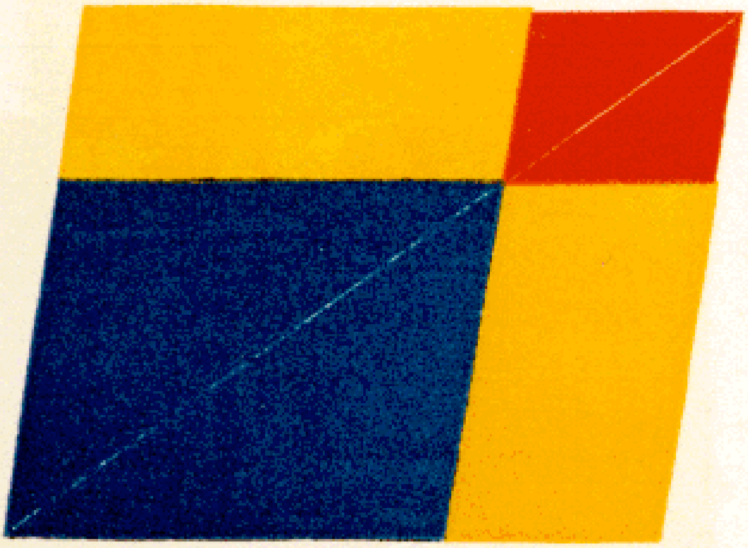
A RECTANGLE or a right angled parallelogram is said to be contained by any two of its adjacent or conterminous sides.

Thus: the right angled parallelogram  is said to be contained by the sides and ; or it may be briefly designated by

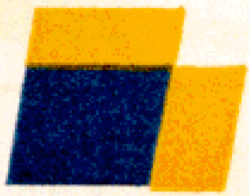
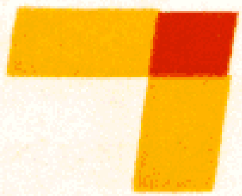
If the adjacent sides are equal; i. e. = , then · which is the expression for the rectangle under and is a square, and

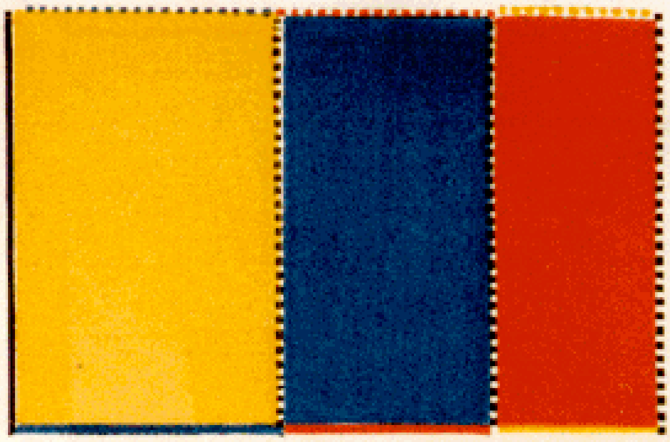


DEFINITION II.

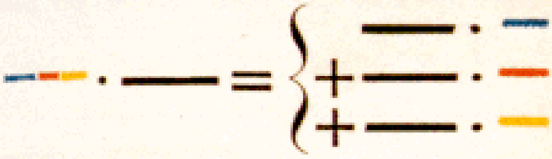


IN a parallelogram, the figure compoſed of one of the parallelograms about the diagonal, together with the two complements, is called a *Gnomon*.

Thus  and  are called Gnomons.



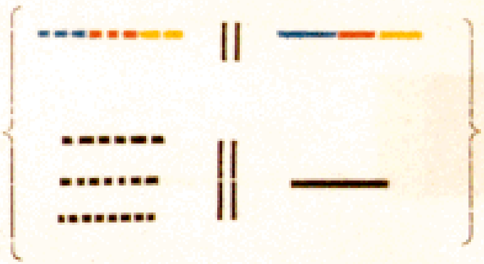
THE *rectangle contained by two straight lines, one of which is divided into any number of parts,*



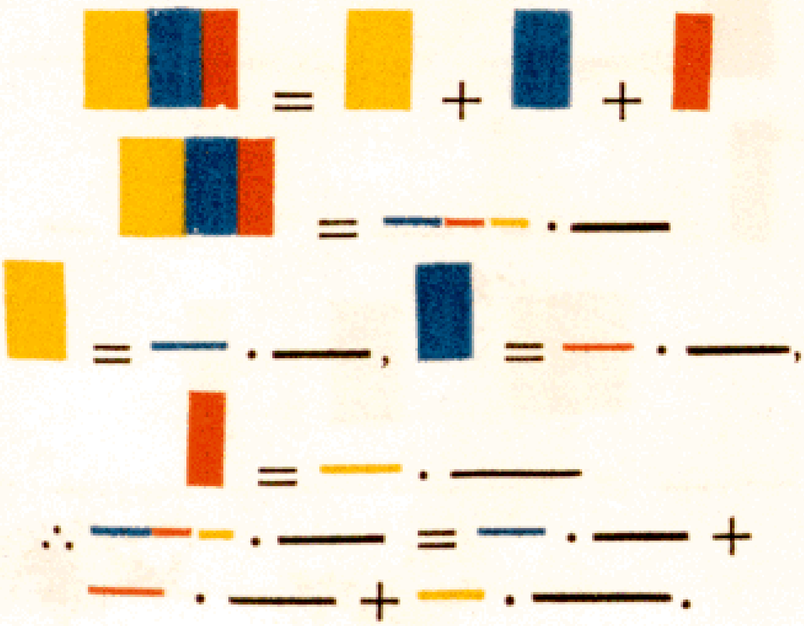
is equal to the sum of the rectangles contained by the undivided line, and the several parts of the divided line.

Draw … ⊥ … and = … (prs. 2. 3. B. 1.); complete the parallelograms, that is to say,

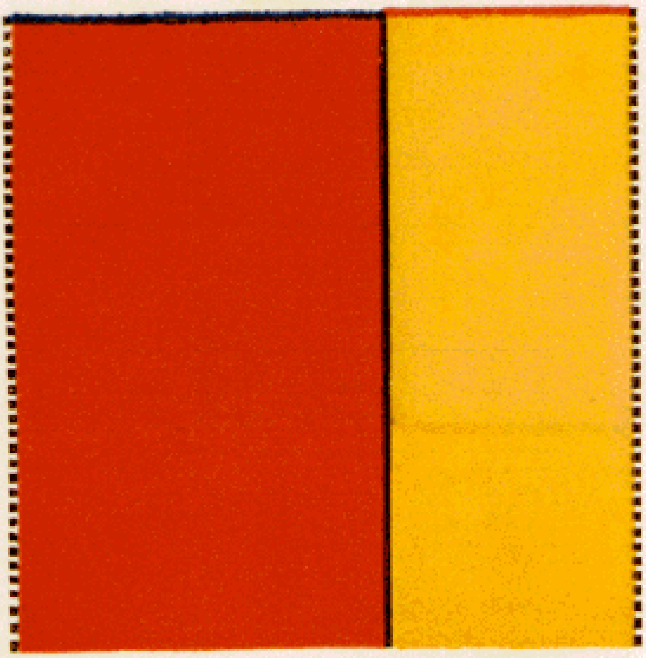
Draw



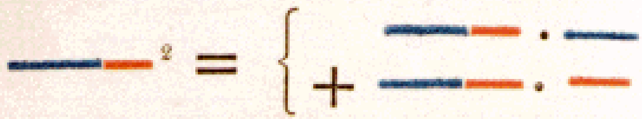
(pr. 31. B. 1.)

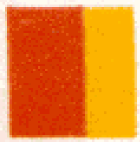


Q. E. D.

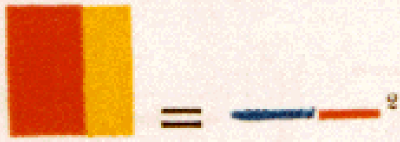


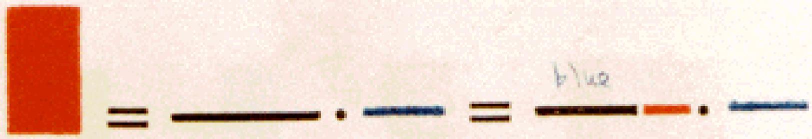
If a straight line be divided into any two parts *, the square of the whole line is equal to the sum of the rectangles contained by the whole line and each of its parts.*

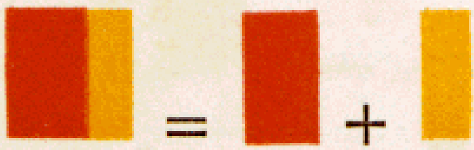


Describe  (B. 1. pr. 46.)

Draw parallel to  (B. 1. pr. 31.)

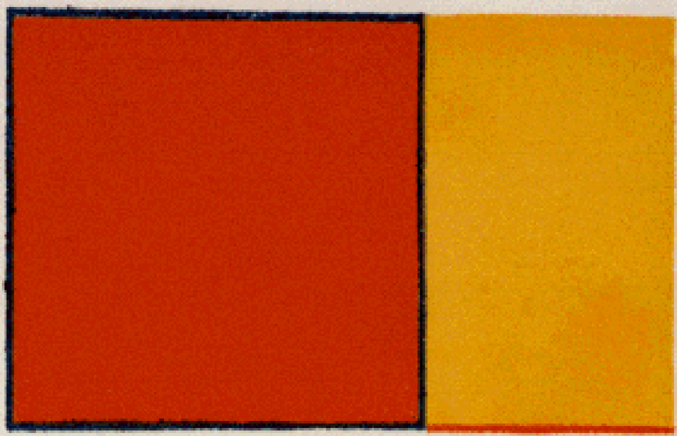




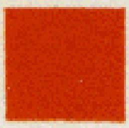


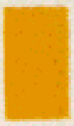
∴ 

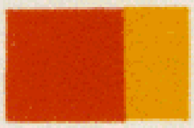
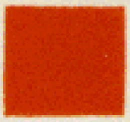
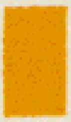
Q. E. D.

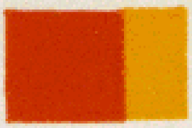


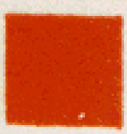
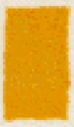
IF a straight line be divided into any two parts *, the rectangle contained by the whole line and either of its parts, is equal to the square of that part, together with the rectangle under the parts.*

Describe  (pr. 46, B. 1.)

Complete  (pr. 31, B. 1.)

Then  =  + , but

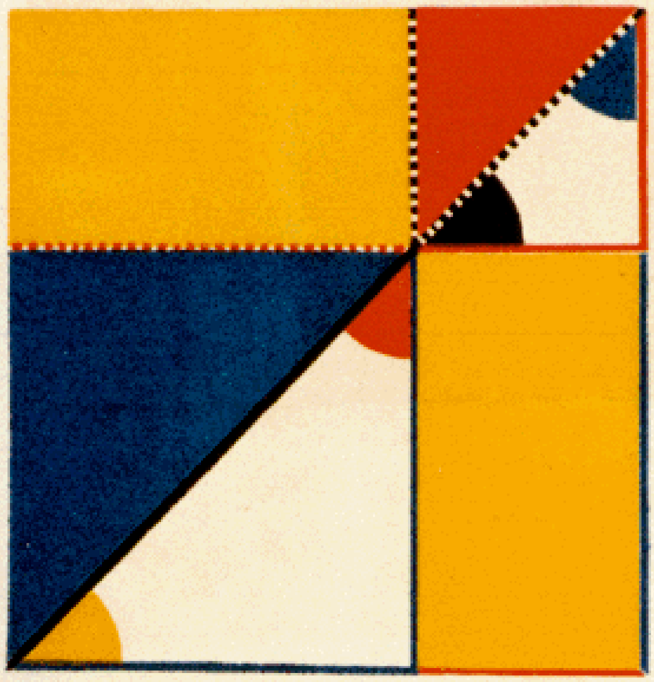
 =  and

 = ,  = ,

∴ 

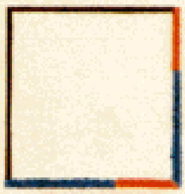
In a similar manner it may be readily shown that 

Q. E. D



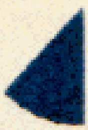
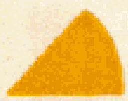
IF a ſtraight line be divided into any two parts , *the ſquare of the whole line is equal to the ſquares of the parts, together with twice the rectangle contained by the parts.*

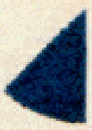
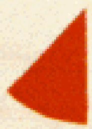
 ^{2} = ^{2} + ^{2} +
twice · .

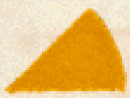
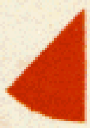
Deſcribe  (pr. 46, B. 1.)

draw  (poſt. 1.),

and {  ||
 || } (pr. 31, B. 1.)

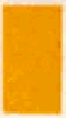
 =  (pr. 5, B. 1.),

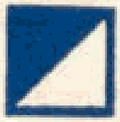
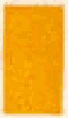
 =  (pr. 29, B. 1.)

∴  = 

∴ by (prs. 6, 29, 34. B. 1.)  is a ſquare = ^{2}.

For the ſame reaſons  is a ſquare = ^{2},

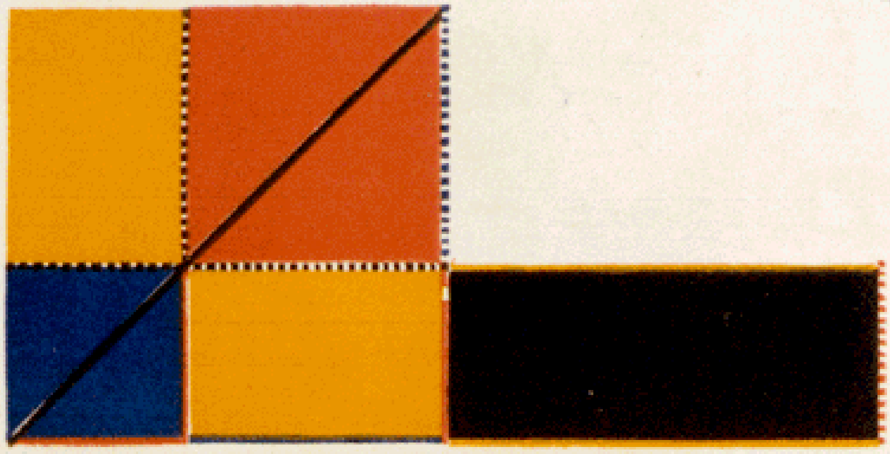
 =  = · (pr. 43, b. 1.)

but  =  +  +  + ,

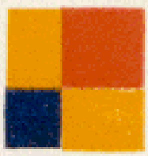
∴ ^{2} = ^{2} + ^{2} +

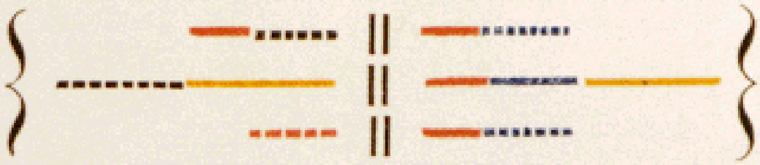
twice · .

Q. E. D.

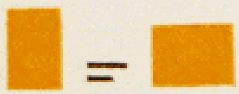


IF a straight line be divided  *into two equal parts and also*  *into two unequal parts, the rectangle contained by the unequal parts, together with the square of the line between the points of section, is equal to the square of half that line*

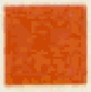
Describe  (pr. 46, B. 1.), draw and

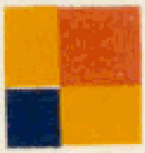
 (pr. 31, B. 1.)

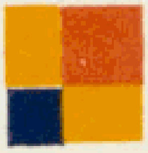
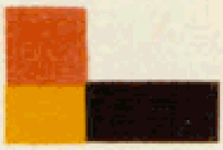
 (p. 36, B. 1.)

 (p. 43, B. 1.)

∴ (ax. 2.) 

but  = ² (cor. pr. 4. B. 2.)

and  = ² (conſt.)

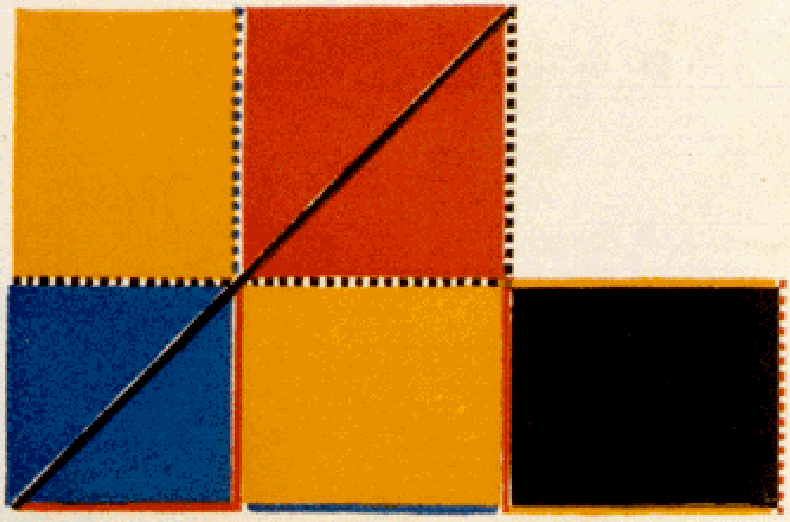
∴ (ax. 2.)  = 

∴ ·  + 

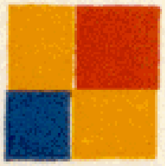
² =

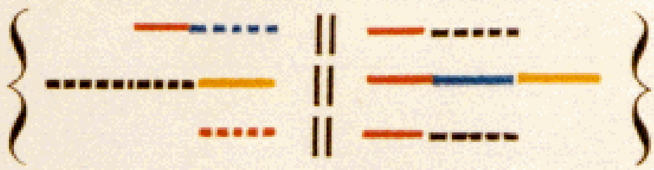
 ² =  ².

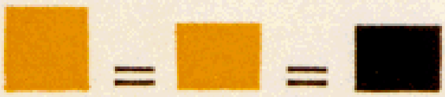
Q. E. D.

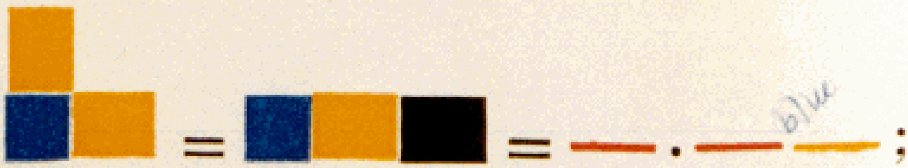


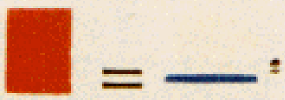
IF a straight line be bisected *and produced to any point* *, the rectangle contained by the whole line so increased, and the part produced, together with the square of half the line, is equal to the square of the line made up of the half, and the produced part.*

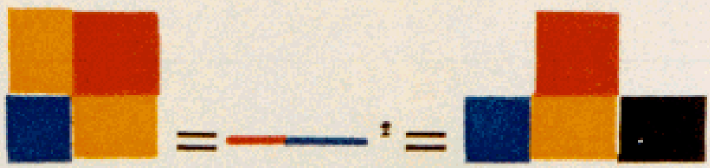
Describe  (pr. 46, B. 1.), draw 

and  (pr. 31, B. 1.)

 (prs. 36, 43, B. 1)

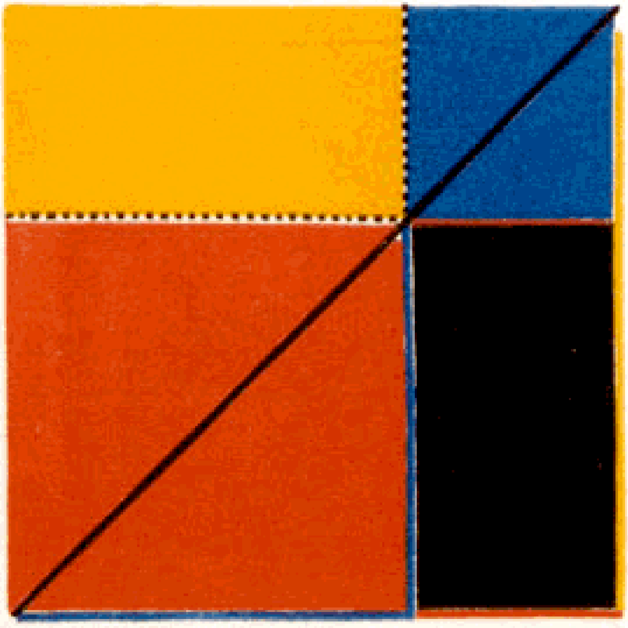
∴ 

but  (cor. 4, B. 2.)

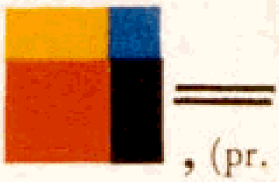
∴  (conſt. ax. 2.)

∴ 

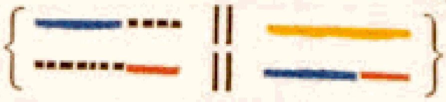
Q. E. D.

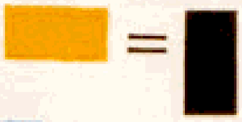


If a ſtraight line be divided into any two parts *, the ſquares of the whole line and one of the parts are equal to twice the rectangle contained by the whole line and that part, together with the ſquare of the other parts.*

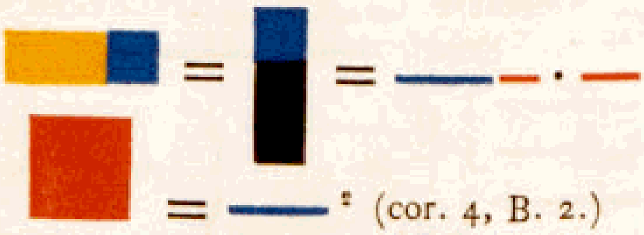
Deſcribe , (pr. 46, B. 1.).

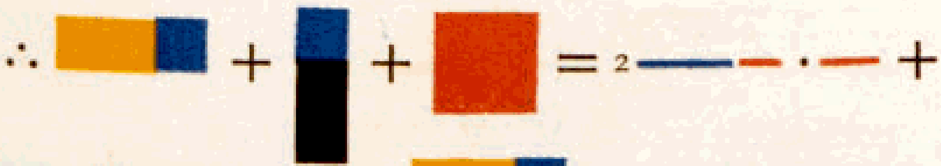
Draw  (poſt. 1.),

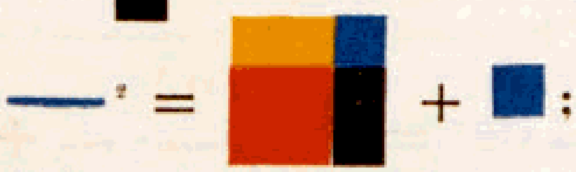
and  (pr. 31, B. 1.).

 (pr. 43, B. 1.),

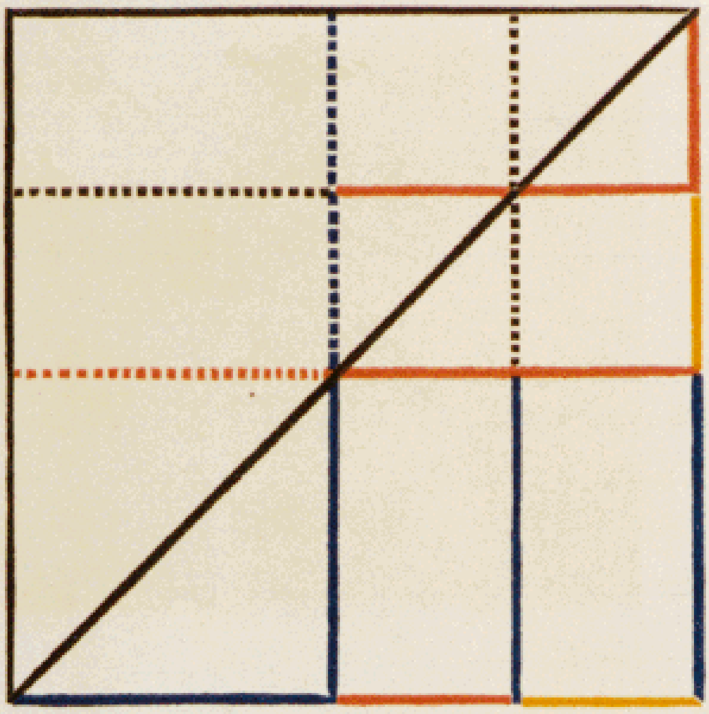
add  to both, (cor. 4, B. 2.)

 (cor. 4, B. 2.)





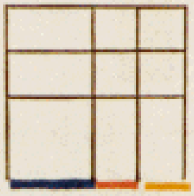
Q. E. D.



IF a straight line be divided into any two parts ——, *the square of the sum of the whole line and any one of its parts, is equal to four times the rectangle contained by the whole line, and that part together with the square of the other part.*

$$—^2 = 4 \cdot — \cdot — + —^2,$$

Produce —— and make — = —

Construct  (pr. 46, B. 1.);

draw ——,

— } ‖ — } (pr. 31, B. 1.)

— } ‖ —

$$—^2 = —^2 + —^2 + 2 \cdot — \cdot —$$

(pr. 4, B. 11.)

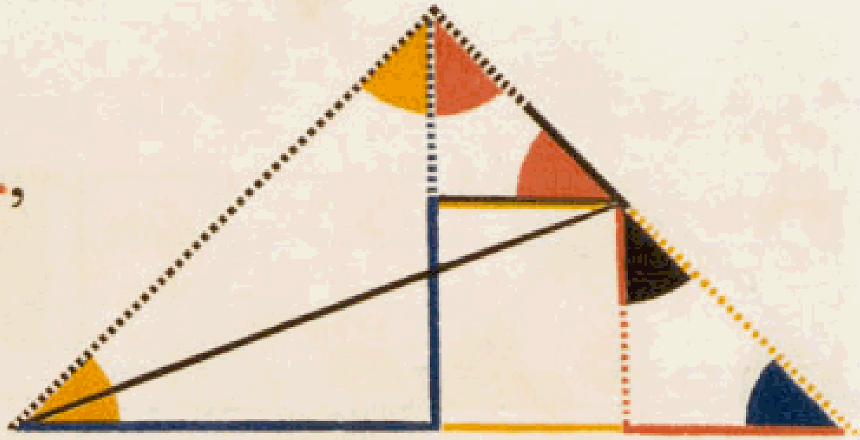
but $—^2 + —^2 = 2 \cdot — \cdot — + —^2$

(pr. 7, B. 11.)

$$\therefore —^2 = 4 \cdot — \cdot — + —^2.$$

Q. E. D.

If a straight line be divided into two equal parts, and also into two unequal parts, the squares of the unequal parts are together double the squares of half the line, and of the part between the points of section.



$^2 + \ ^2 = 2\ ^2 + 2\ ^2.$

Make ⊥ and = or ,

Draw and ,

|| , || , and draw .

= (pr. 5, B. 1.) = half a right angle. (cor. pr. 32, B. 1.)

= (pr. 5, B. 1.) = half a right angle. (cor. pr. 32, B. 1.)

∴ = a right angle.

= = =

(prs. 5, 29, B. 1.).

hence = , = =

(prs. 6, 34, B. 1.)

^{2} = { $^2 + \ ^2$, or + ^{2}

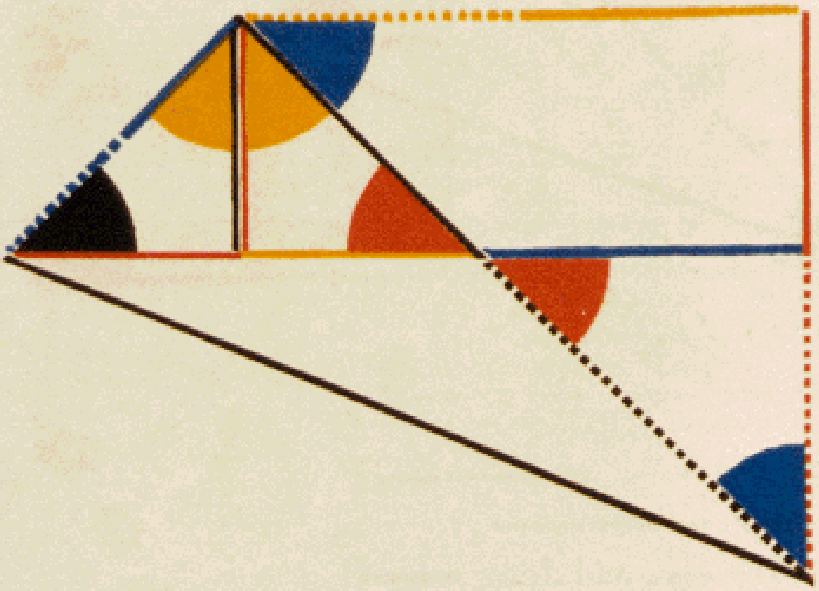
= { $^2 = 2\ ^2$

(pr. 47, B. 1.)

$^2 = 2\ ^2$ }

∴ $^2 + \ ^2 = 2\ ^2 + 2\ ^2.$

Q. E. D.



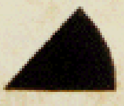
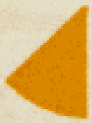
IF a straight line  *be bisected and produced to any point* , *the squares of the whole produced line, and of the produced part, are together double of the squares of the half line, and of the line made up of the half and produced part.*

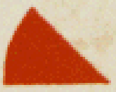
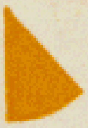
Make  ⊥ and = to or ,

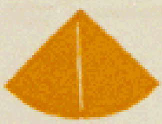
draw  and ,

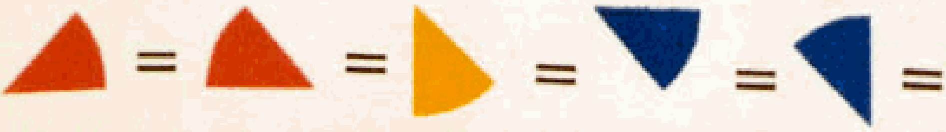
and { ||  ;  ||  } (pr. 31, B. 1.);

draw  also.

 =  (pr. 5, B. 1.) = half a right angle. (cor. pr. 32, B. 1.)

 =  (pr. 5, B. 1.) = half a right angle (cor. pr. 32, B. 1.)

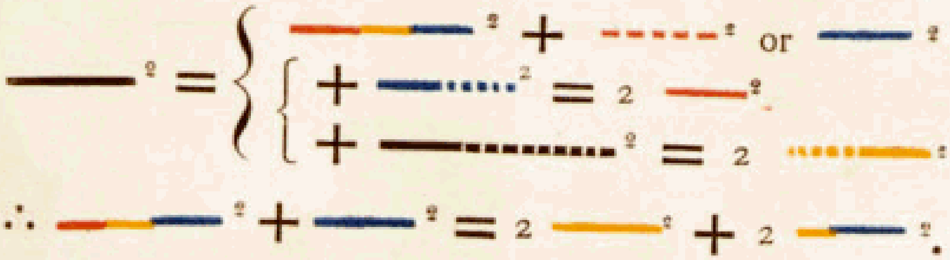
∴  = a right angle.



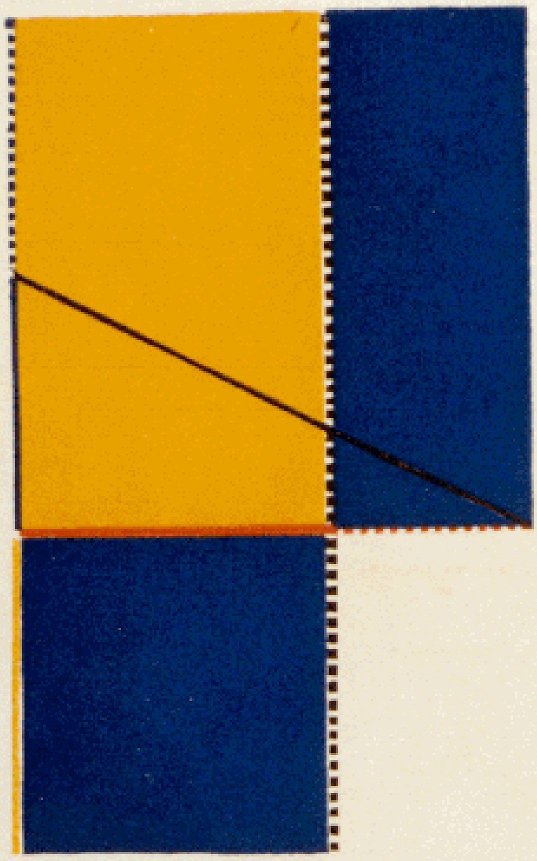
half a right angle (prs. 5, 32, 29, 34, B. 1.),

and 

, (prs. 6, 34, B. 1.). Hence by (pr. 47, B. 1.)



Q. E. D.



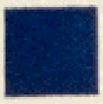
To divide a given straight line  *in such a manner, that the rectangle contained by the whole line and one of its parts may be equal to the square of the other.*

Describe  (pr. 46, B. 1.),

make = (pr. 10, B. 1.),

draw ,

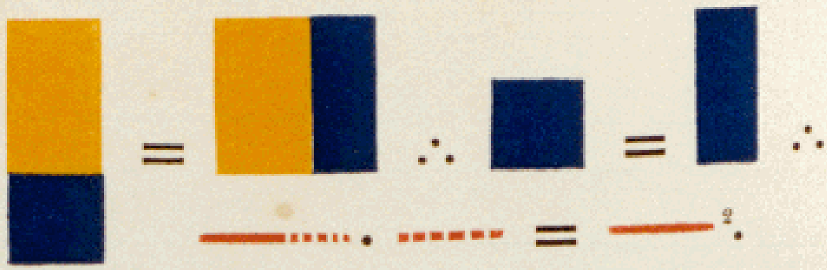
take = (pr. 3, B. 1.),

on describe  (pr. 46, B. 1.),

Produce (post. 2.).

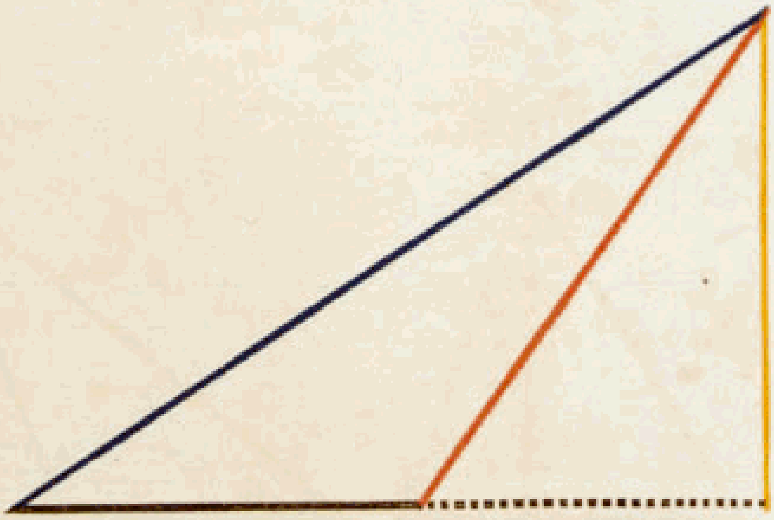
Then, (pr. 6, B. 2.)  +

 , or,



Q. E. D.

In any obtuse angled triangle, the square of the side subtending the obtuse angle exceeds the sum of the squares of the sides containing the obtuse angle, by twice the rectangle contained by either of these sides and the produced parts of the same from the obtuse angle to the perpendicular let fall on it from the opposite acute angle.



[blue line]^2 ⊏ [black line]^2 + [red line]^2 by 2 [black line] · [dotted line].

By pr. 4, B. 2.

[black and dotted line]^2 = [black line]^2 + [dotted line]^2 + 2 [black line] · [dotted line]:

add [yellow line]^2 to both

[black and dotted line]^2 + [yellow line]^2 = [blue line]^2 (pr. 47, B. 1.)

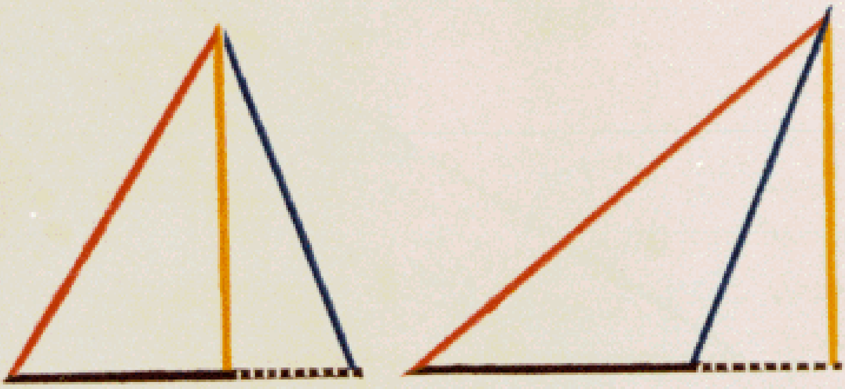
= 2 · [black line] · [dotted line] + [black line]^2 + { [dotted line]^2 + [yellow line]^2 } or

+ [red line]^2 (pr. 47, B. 1.). Therefore,

[blue line]^2 = 2 · [black line] · [dotted line] + [black line]^2 + [red line]^2: hence [blue line]^2 ⊏ [red line]^2 + [black line]^2

by 2 · [black line] · [dotted line].

Q. E. D.

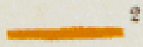


IN any triangle, the ſquare of the ſide ſubtending an acute angle, is leſs than the ſum of the ſquares of the ſides containing that angle, by twice the rectangle contained by either of theſe ſides, and the part of it intercepted between the foot of the perpendicular let fall on it from the oppoſite angle, and the angular point of the acute angle.

FIRST.

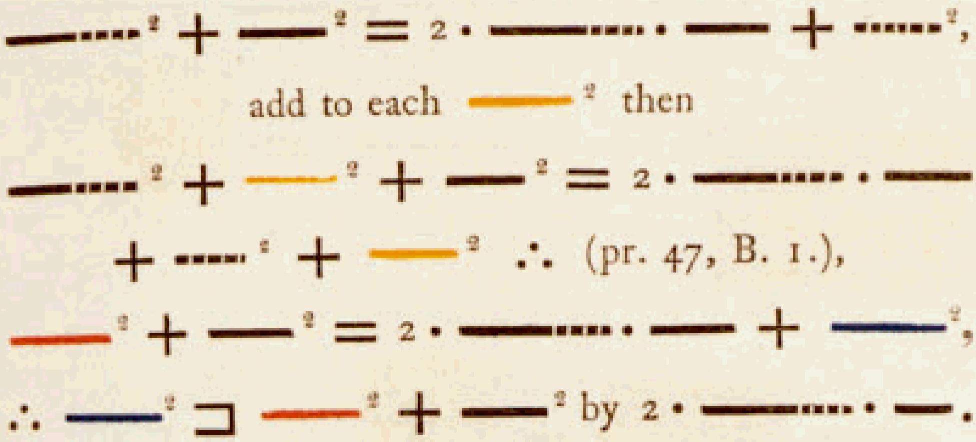
SECOND.

Firſt, ſuppoſe the perpendicular to fall within the triangle, then (pr. 7, B. 2.)

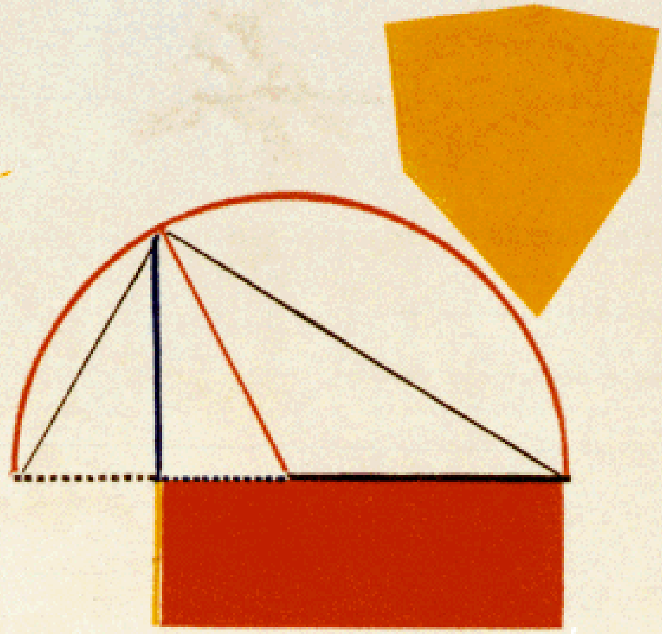
add to each  then,

∴ (pr. 47, B. 1.)

Next suppose the perpendicular to fall without the triangle, then (pr. 7, B. 2.)

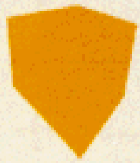


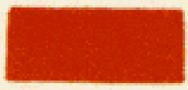
Q. E. D.



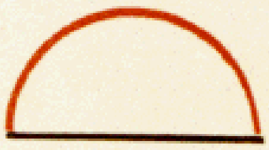
TO *draw a right line of which the square shall be equal to a given rectilinear figure.*

To draw *such that,*

^{2} = 

Make  =  (pr. 45, B. 1.),

produce until = ;
take = (pr. 10, B. 1.),

Describe  (post. 3.),
and produce to meet it : draw .
^{2} or ^{2} = · + ^{2}
(pr. 5, B. 2.),
but ^{2} = ^{2} + ^{2} (pr. 47, B. 1.);
∴ ^{2} + ^{2} = · + ^{2},
∴ ^{2} = · , and

∴ ^{2} =  = 

Q. E. D.

BOOK III.

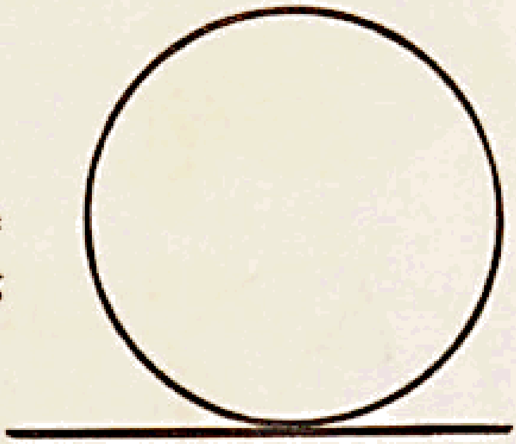
DEFINITIONS.

I.

EQUAL circles are thoſe whoſe diameters are equal.

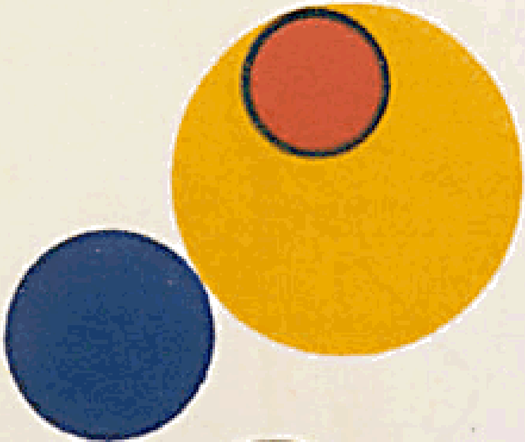
II.

A right line is said to touch a circle when it meets the circle, and being produced does not cut it.



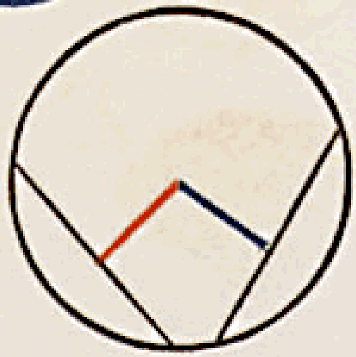
III.

Circles are ſaid to touch one another which meet but do not cut one another.



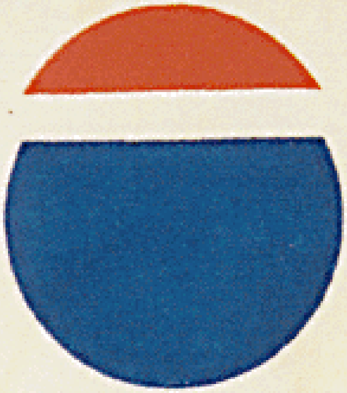
IV.

Right lines are ſaid to be equally diſtant from the centre of a circle when the perpendiculars drawn to them from the centre are equal.



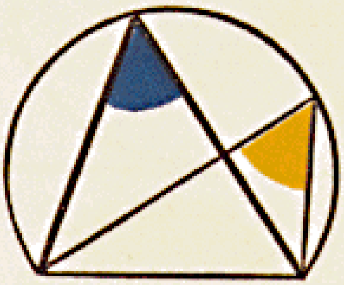
V.

And the ſtraight line on which the greater perpendicular falls is ſaid to be farther from the centre.



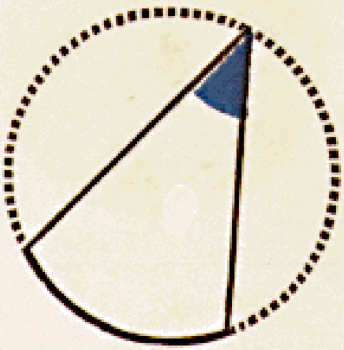
VI.

A ſegment of a circle is the figure contained by a ſtraight line and the part of the circumference it cuts off.



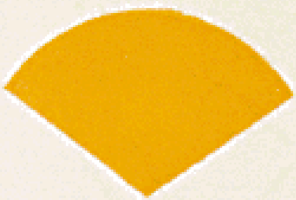
VII.

An angle in a ſegment is the angle contained by two ſtraight lines drawn from any point in the circumference of the ſegment to the extremities of the ſtraight line which is the baſe of the ſegment.



VIII.

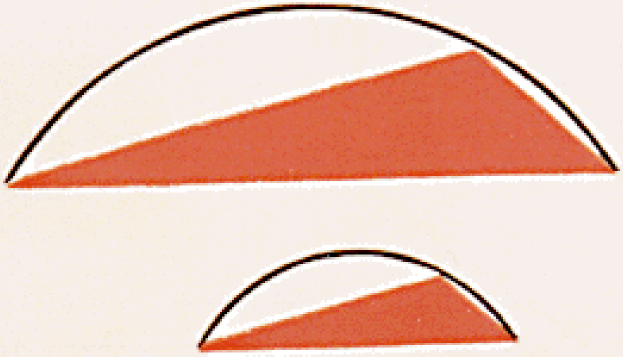
An angle is ſaid to ſtand on the part of the circumference, or the arch, intercepted between the right lines that contain the angle.



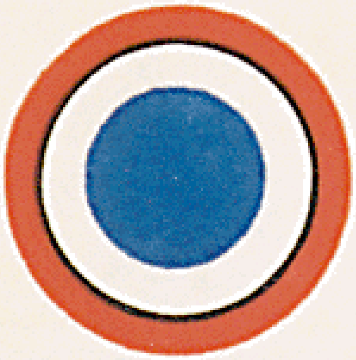
IX.

A ſector of a circle is the figure contained by two radii and the arch between them.

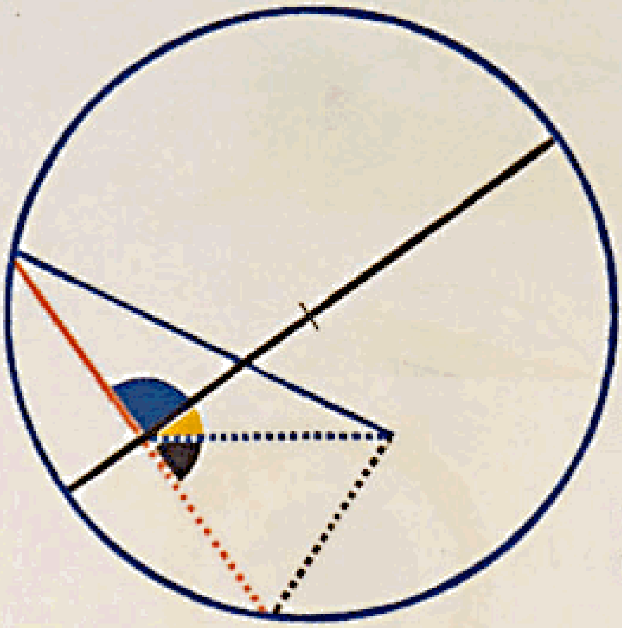
X.

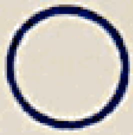


Similar ſegments of circles are thoſe which contain equal angles.



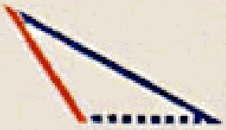
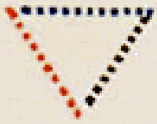
Circles which have the ſame centre are called *concentric circles*.

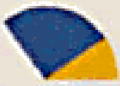
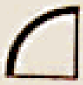


TO *find the centre of a given circle* .

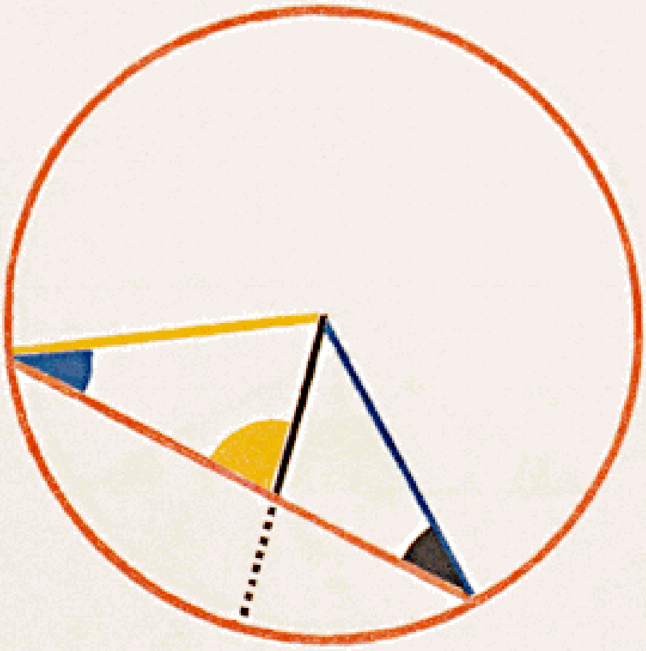
Draw within the circle any ſtraight line , make = , draw ⊥ ; biſect , and the point of biſection is the centre.

For, if it be poſſible, let any other point as the point of concourſe of , and be the centre.

Becauſe in  and 

 = (hyp. and B. 1, def. 15.)
 =  (conſt.) and common,
 =  (B. 1, pr. 8.), and are therefore right angles; but  =  (conſt.)  =  (ax. 11.) which is abſurd; therefore the aſſumed point is not the centre of the circle; and in the ſame manner it can be proved that no other point which is not on is the centre, therefore the centre is in , and therefore the point where  is biſected is the centre.

Q. E. D.



A STRAIGHT *line* ( ) *joining two points in the circumference of a circle* , *lies wholly within the circle.*

Find the centre of (B.3.pr.1.);

from the centre draw to any point in ,

meeting the circumference from the centre ;

draw and .

Then = (B. 1. pr. 5.)

but ⊏ or ⊏ (B. 1. pr. 16.)

∴ ⊏ (B. 1. pr. 19.)

but = ,

∴ ⊏ ;

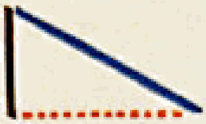
∴ ⊐ ;

∴ every point in lies within the circle.

Q. E. D.

IF a straight line ( ) drawn through the centre of a circle bisects a chord ( ) which does not pass through the centre, it is perpendicular to it; or, if perpendicular to it, it bisects it.

Draw and to the centre of the circle.

In  and 

= , common, and

= ∴ = (B. 1. pr. 8.)

and ∴ ⊥ (B. 1. def. 7.)

Again let ⊥

Then in  and 

= (B. 1. pr. 5.)

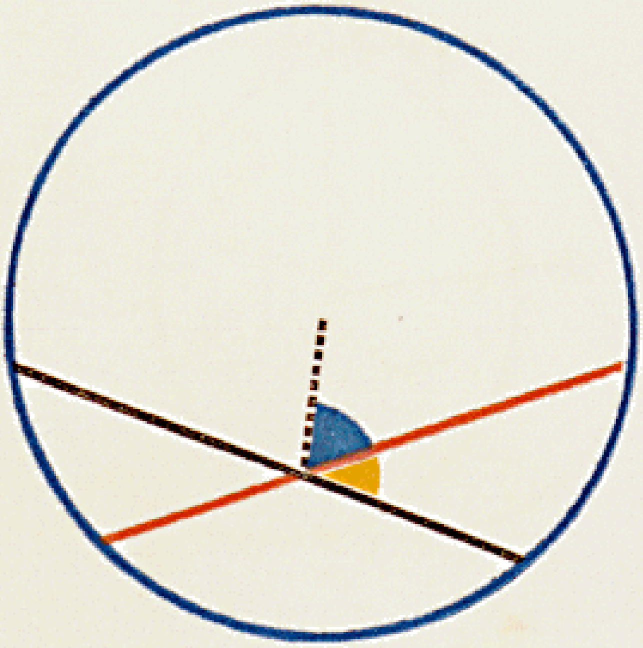
= (hyp.)

and =

∴ = (B. 1. pr. 26.)

and ∴ bisects .

Q. E. D.

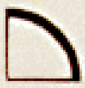


IF in a circle two ſtraight lines cut one another, which do not both paſs through the centre, they do not biſect one another.

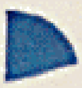
If one of the lines paſs through the centre, it is evident that it cannot be biſected by the other, which does not paſs through the centre.

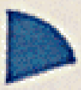
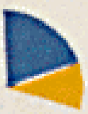
But if neither of the lines or
paſs through the centre, draw
from the centre to their interſection.

If be biſected, ⊥ to it (B. 3. pr. 3.)

∴  =  and if be

biſected, ⊥ (B. 3. pr. 3.)

∴  = 

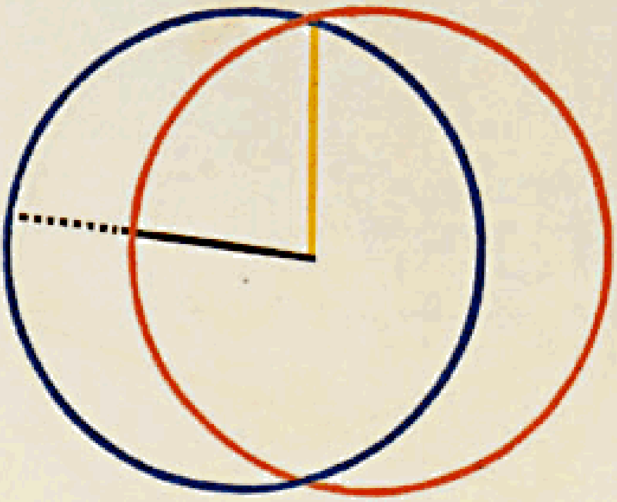
and ∴  = ; a part

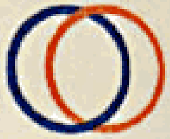
equal to the whole, which is abſurd :

∴ and

do not biſect one another.

Q. E. D.



IF *two circles*  *interfect, they have not the fame centre.*

Suppofe it poffible that two interfecting circles have a common centre; from fuch fuppofed centre draw to the interfecting point, and  meeting the circumferences of the circles.

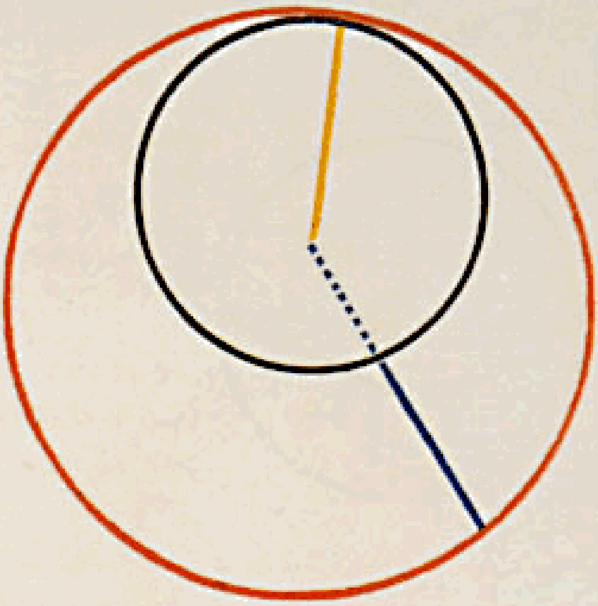
Then =  (B. 1. def. 15.)

and =  (B. 1. def. 15.)

∴ = ; a part equal to the whole, which is abfurd:

∴ circles fuppofed to interfect in any point cannot have the fame centre.

Q. E. D.



IF *two circles*  *touch one another internally, they have not the same centre.*

For, if it be possible, let both circles have the same centre; from such a supposed centre draw cutting both circles, and to the point of contact.

Then =  (B. 1. def. 15.)

and = (B. 1. def. 15.)

∴  = ; a part equal to the whole, which is absurd;

therefore the assumed point is not the centre of both circles; and in the same manner it can be demonstrated that no other point is.

Q. E. D.

FIGURE I.

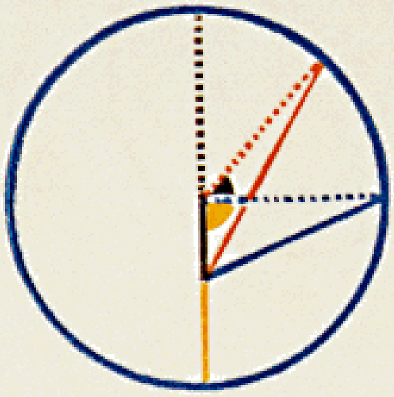
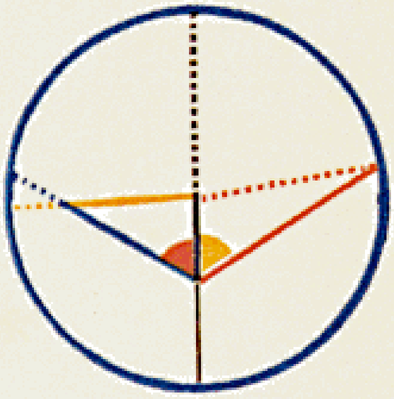


FIGURE II.



IF from any point within a circle which is not the centre, lines are drawn to the circumference; the greatest of those lines is that ( ) which passes through the centre, and the least is the remaining part ( ) of the diameter.

Of the others, that ( ) which is nearer to the line passing through the centre, is greater than that ( ) which is more remote.

Fig. 2. *The two lines (and) which make equal angles with that passing through the centre, on opposite sides of it, are equal to each other; and there cannot be drawn a third line equal to them, from the same point to the circumference.*

FIGURE I.

To the centre of the circle draw and ;

then = (B. 1. def. 15.)

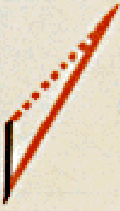
= + ⊏ (B. 1. pr. 20.)

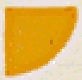
in like manner may be shewn to be greater than , or any other line drawn from the same point to the circumference. Again, by (B. 1. pr. 20.)

+ ⊏ = + ,

take from both; ∴ ⊏ (ax.),

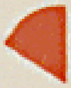
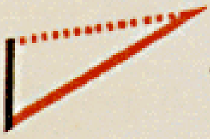
and in like manner it may be shewn that is less than any other line drawn from the ſame point to the cir-

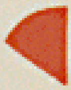
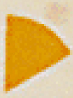
cumference. Again, in  and ,

 common,  ⊏ , and =

∴  ⊏ (B. 1. pr. 24.) and  may in like manner be proved greater than any other line drawn from the ſame point to the circumference more remote from .

FIGURE II.

If  =  then = , if not take  = draw , then in  and , common,

 =  and  = 

∴  = (B. 1. pr. 4.)

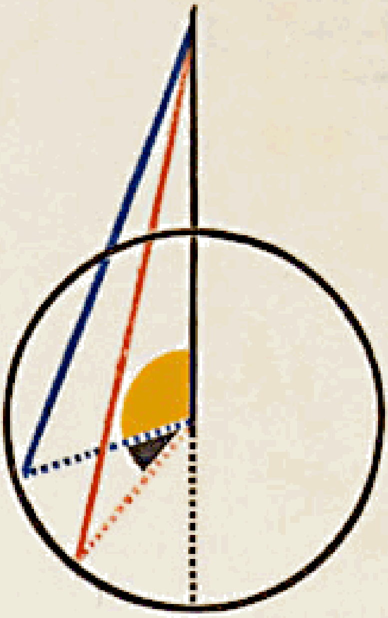
∴  = =

a part equal to the whole, which is abſurd:

∴  = ; and no other line is equal to  drawn from the ſame point to the circumference; for if it were nearer to the one paſſing through the centre it would be greater, and if it were more remote it would be leſs.

Q. E. D.

The original text of this propoſition is here divided into three parts.

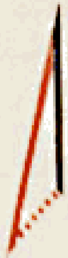


I.

IF from a point without a circle, ſtraight lines { &c. } are drawn to the circumference; of thoſe falling upon the concave circumference the greateſt is that ( ) which paſſes through the centre, and the line ( ) which is nearer the greateſt is greater than that ( ) which is more remote.

Draw and to the centre.

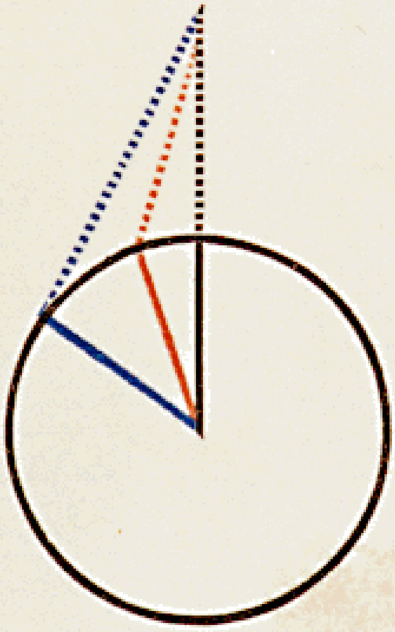
Then, which paſſes through the centre, is greateſt; for ſince = , if be added to both, = + ; but ⊏ (B. 1. pr. 20.) ∴ is greater than any other line drawn from the ſame point to the concave circumference.

Again in  and , = , and common, but ⊏ ,

∴ ⊏ (B. 1. pr. 24.);

and in like manner may be ſhewn ⊏ than any other line more remote from .

II.



Of thoſe lines falling on the convex circumference the leaſt is that ( ) which being produced would paſs through the centre, and the line which is nearer to the leaſt is leſs than that which is more remote.

For, ſince + ⊏ (B. 1. pr. 20.)

and = ,

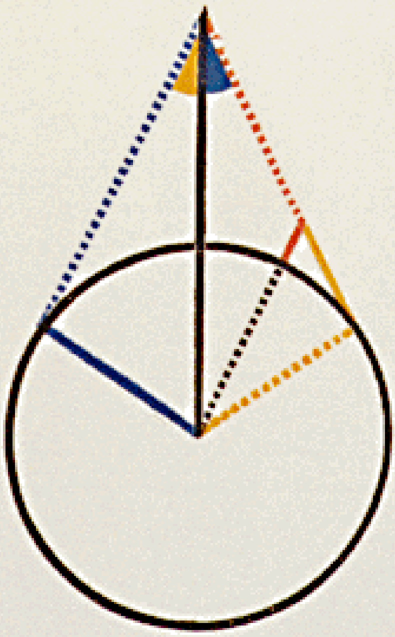
∴ ⊏ (ax. 5.)

And again, ſince + ⊏ + (B. 1. pr. 21.),

and = ,

∴ ⊐ . And ſo of others.

III.



Alſo the lines making equal angles with that which paſſes through the centre are equal, whether falling on the concave or convex circumference; and no third line can be drawn equal to them from the ſame point to the circumference.

For if ⊏ , but making = ;

make = , and draw .

Then in  and  we have ——— = ———,

and ——— common, and alſo  = ,

∴ ——— = ——— (B. 1. pr. 4.);

but ——— = ———;

∴ ——— = ———, which is abſurd.

∴ ——— is not = ———, nor to any part

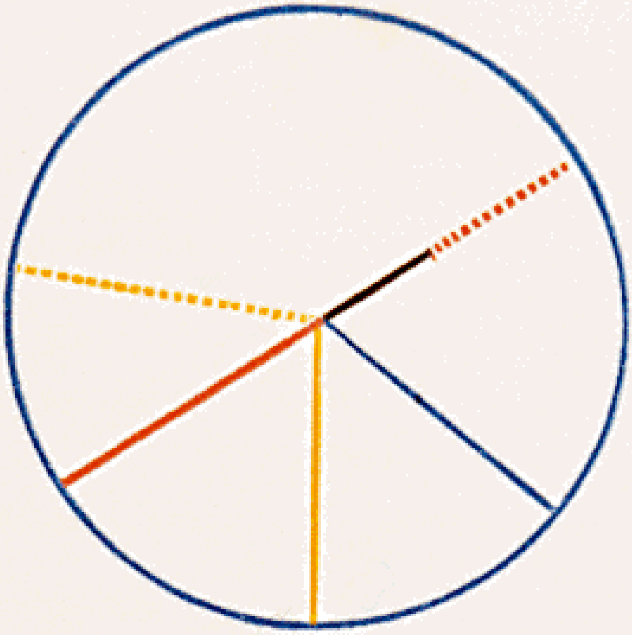
of ———, ∴ ——— is not ⊏ ———.

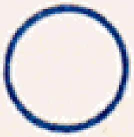
Neither is ——— ⊏ ———, they are

∴ = to each other.

And any other line drawn from the ſame point to the circumference muſt lie at the ſame ſide with one of theſe lines, and be more or leſs remote than it from the line paſſing through the centre, and cannot therefore be equal to it.

Q. E. D.



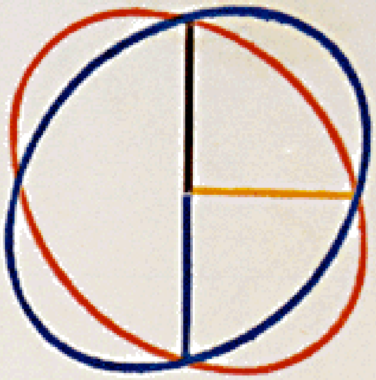
IF a point be taken within a circle  *, from which more than two equal ſtraight lines (, ,) can be drawn to the circumference, that point muſt be the centre of the circle.*

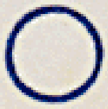
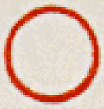
For, if it be ſuppoſed that the point in which more than two equal ſtraight lines meet is not the centre, ſome other point muſt be; join theſe two points by , and produce it both ways to the circumference.

Then ſince more than two equal ſtraight lines are drawn from a point which is not the centre, to the circumference, two of them at leaſt muſt lie at the ſame ſide of the diameter ; and ſince from a point , which is not the centre, ſtraight lines are drawn to the circumference; the greateſt is , which paſſes through the centre: and which is nearer to , ⊏ which is more remote (B. 3. pr. 8.); but = (hyp.) which is abſurd.

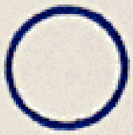
The ſame may be demonſtrated of any other point, different from , which muſt be the centre of the circle.

Q. E. D.



ONE *circle*  *cannot interfect another*  *in more points than two.*

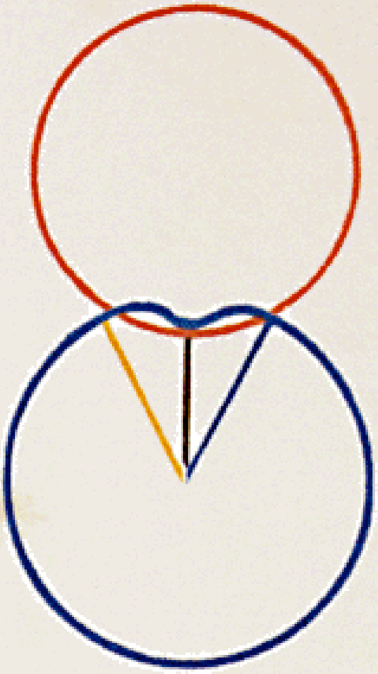
For, if it be poffible, let it interfect in three points;

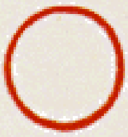
from the centre of  draw ,  and to the points of interfection;

∴  = =

(B. 1. def. 15.),

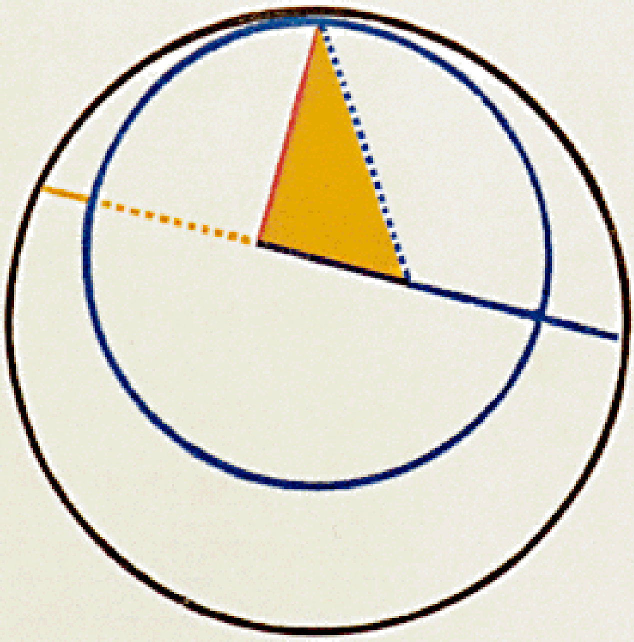
but as the circles interfect, they have not the fame centre (B. 3. pr. 5.):

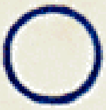
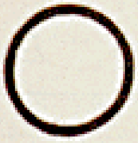


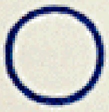
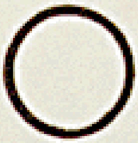
∴ the affumed point is not the centre of , and

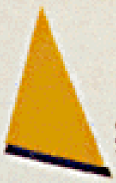
∴ as , and are drawn from a point not the centre, they are not equal (B. 3. prs. 7, 8); but it was fhewn before that they were equal, which is abfurd; the circles therefore do not interfect in three points.

Q. E. D.



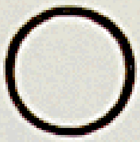
IF *two circles*  *and*  *touch one another internally, the right line joining their centres, being produced, shall pass through a point of contact.*

For, if it be possible, let join their centres, and produce it both ways; from a point of contact draw to the centre of , and from the same point of contact draw to the centre of .

Because in ; + ⊏

(B. 1. pr. 20.),

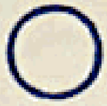
and = as they are radii of

,

but + ⊏ ; take away which is common,

and ⊏ ;

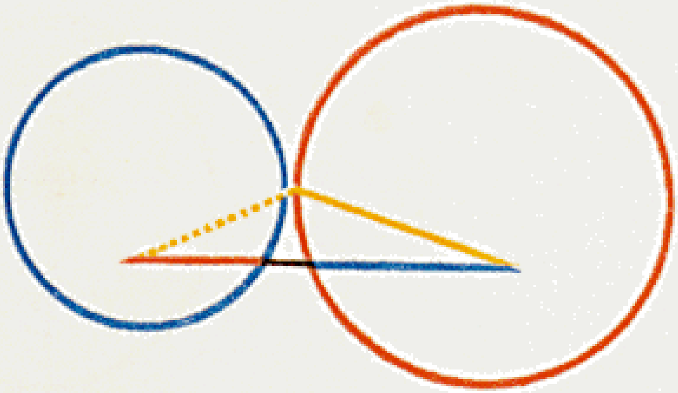
but = ,

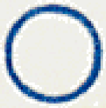
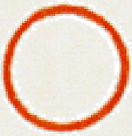
becauſe they are radii of ,

and ∴ ⊏ a part greater than the whole, which is abſurd.

The centres are not therefore ſo placed, that a line joining them can paſs through any point but a point of contact.

Q. E. D.



IF two circles  *and*  *touch one another externally, the straight line*  *joining their centres, passes through the point of contact.*

If it be possible, let  join the centres, and not pass through a point of contact; then from a point of contact draw and to the centres.

Because + ⊏ (B. 1. pr. 20.),

and = (B. 1. def. 15.),

and = (B. 1. def. 15.),

∴ + ⊏ , a part greater than the whole, which is absurd.

The centres are not therefore so placed, that the line joining them can pass through any point but the point of contact.

Q. E. D.

FIGURE I.

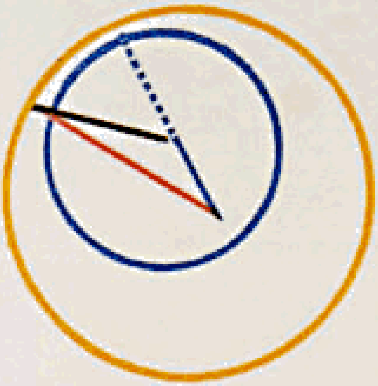
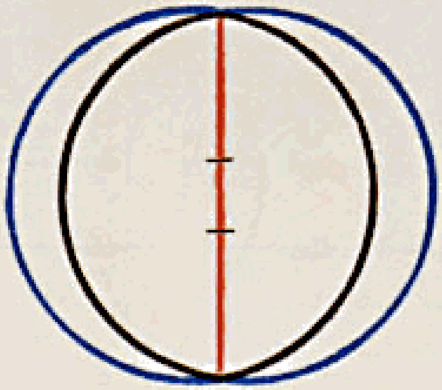


FIGURE II.



ONE circle cannot touch another, either externally or internally, in more points than one.

FIGURE III.

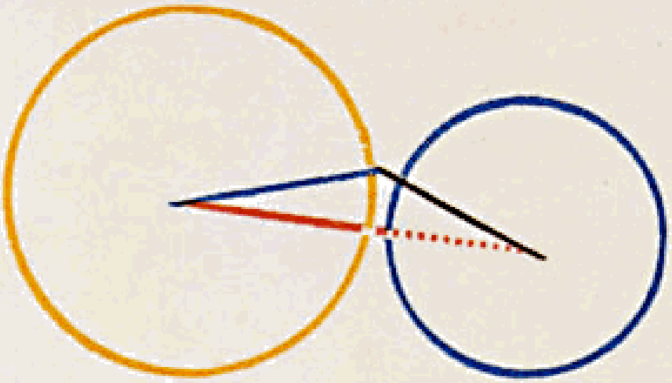
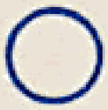


Fig. 1. For, if it be poſſible, let

 and  touch one

another internally in two points; draw joining their centres, and produce it until it paſs through one of the points of contact (B. 3. pr. 11.);

draw and ,

But = (B. 1. def. 15.),

∴ if be added to both,

= + ;

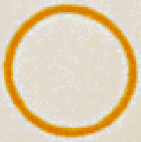
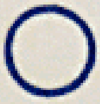
but = (B. 1. def. 15.),

and ∴ + = ; but

+ ⊏ (B. 1. pr. 20.),

which is abſurd.

Fig. 2. But if the points of contact be the extremities of the right line joining the centres, this ſtraight line muſt be biſected in two different points for the two centres; becauſe it is the diameter of both circles, which is abſurd.

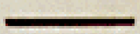
Fig. 3. Next, if it be poſſible, let  and 

touch externally in two points; draw  joining the centres of the circles, and paſſing through one of the points of contact, and draw and .

 = (B. 1. def. 15.);

and  =  (B. 1. def. 15.):

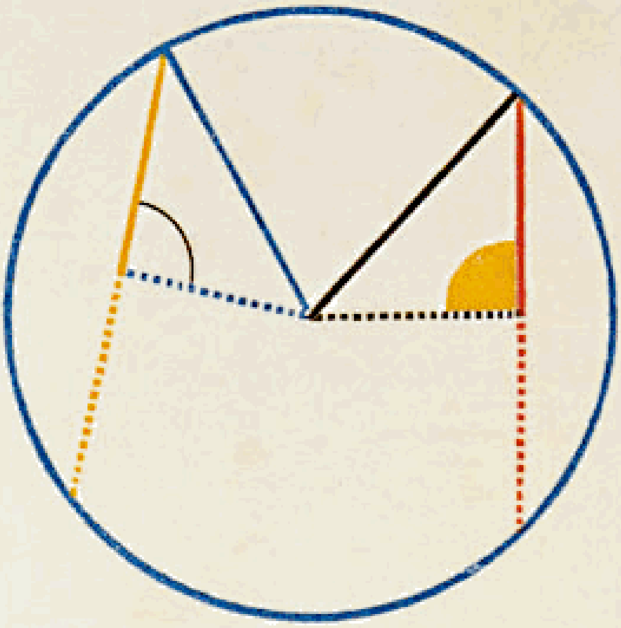
∴ + = ; but

 +  $\sqsubset$  (B. 1. pr. 20.),

which is abſurd.

There is therefore no caſe in which two circles can touch one another in two points.

Q. E. D.



EQUAL ſtraight lines ( ) inſcribed in a circle are equally diſtant from the centre; and alſo, ſtraight lines equally diſtant from the centre are equal.

From the centre of draw

⊥ to and

⊥ , join and .

Then = half (B. 3. pr. 3.)

and $= \frac{1}{2}$ (B. 3. pr. 3.)

ſince = (hyp.)

∴ = ,

and = (B. 1. def. 15.)

∴ $^2 =$ ^{2};

but ſince is a right angle

$^2 =$ $^2 +$ ^{2} (B. 1. pr. 47.)

and $^2 =$ $^2 +$ ^{2} for the ſame reaſon,

∴ $^2 +$ $^2 =$ $^2 +$ ^{2}

∴ ^{2} = ^{2},

∴ = .

Alſo, if the lines and be equally diſtant from the centre; that is to ſay, if the perpendiculars  and be given equal, then = .

For, as in the preceding caſe,

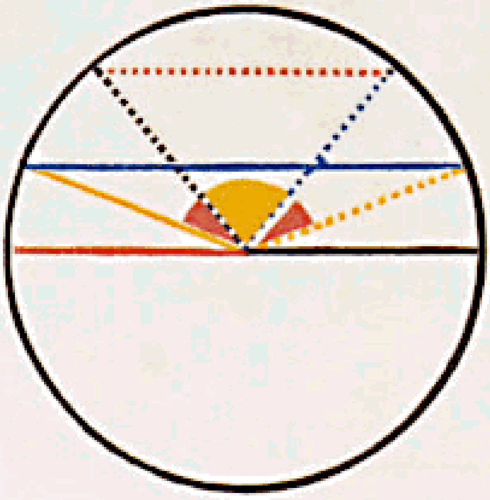
 ^{2} + ^{2} = ^{2} + ^{2};

but ^{2} = ^{2}:

∴ ^{2} = ^{2}, and the doubles of theſe  and are alſo equal.

Q. E. D.

FIGURE I.



THE diameter is the greateſt ſtraight line in a circle: and, of all others, that which is neareſt to the centre is greater than the more remote.

FIGURE I.

The diameter —— is ⊏ any line —.

For draw — and —.

Then — = —

and — = —,

∴ — + — = ——

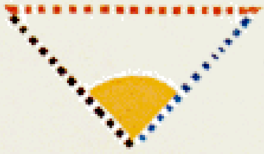
but — + — ⊏ — (B. 1. pr. 20.)

∴ —— ⊏ —.

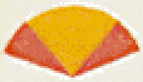
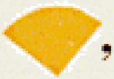
Again, the line which is nearer the centre is greater than the one more remote.

Firſt, let the given lines be — and —, which are at the ſame ſide of the centre and do not interſect;

draw { —, —, —, —.

In  and ,

and = and ;

but  ⊏ ,

∴ ⊏ (B. 1. pr. 24.)

FIGURE II.

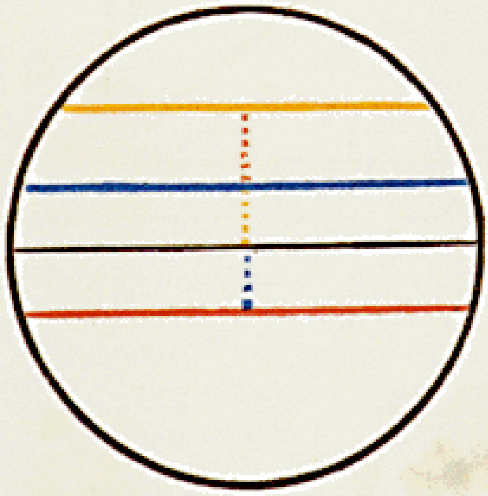


FIGURE II.

Let the given lines be and which either are at different fides of the centre, or interfect; from the centre draw and ⊥ and ,

make = , and

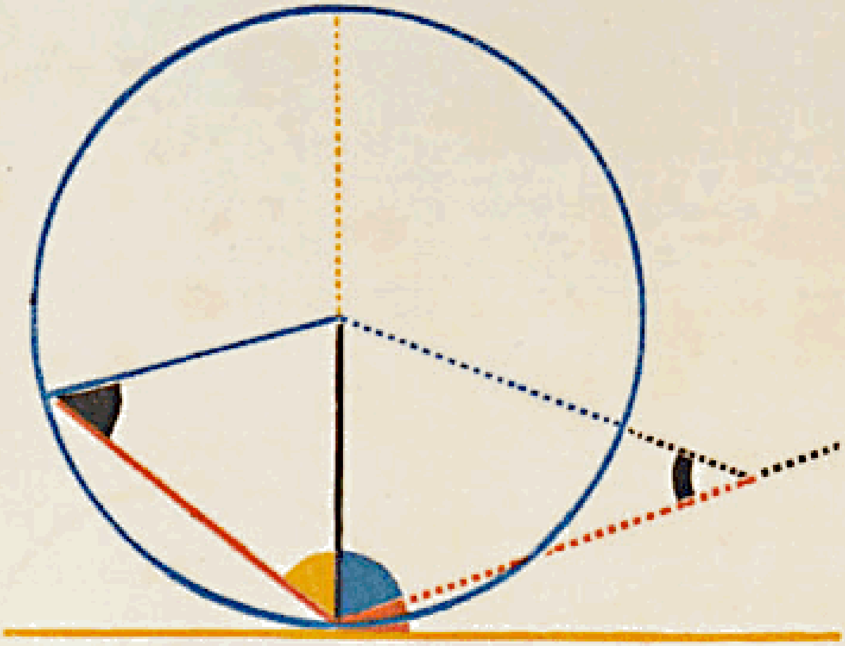
draw ⊥ .

Since and are equally diftant from the centre, = (B. 3. pr. 14.);

but ⊏ (Pt. 1. B. 3. pr. 15.),

∴ ⊏ .

Q. E. D.



THE straight line ——— *drawn from the extremity of the diameter* ——— *of a circle perpendicular to it falls without the circle.*

And if any straight line - - - - - - *be drawn from a point within that perpendicular to the point of contact, it cuts the circle.*

PART I

If it be possible, let ———, which meets the circle again, be ⊥ ———, and draw ———.

Then, because ——— = ———,

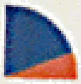
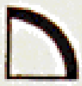
 =  (B. 1. pr. 5.),

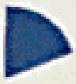
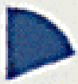
and ∴ each of these angles is acute. (B. 1. pr. 17.)

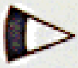
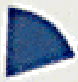
but  =  (hyp.), which is absurd, therefore ——— drawn ⊥ ——— does not meet the circle again.

PART II.

Let be ⊥ and let be drawn from a point between and the circle, which, if it be poſſible, does not cut the circle.

Becauſe  = ,

∴  is an acute angle; ſuppoſe ⊥ , drawn from the centre of the circle, it muſt fall at the ſide of  the acute angle.

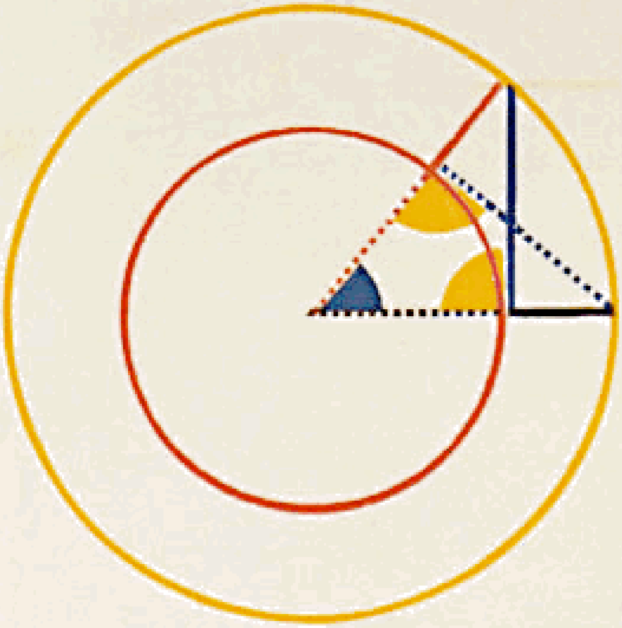
∴  which is ſuppoſed to be a right angle, is ⊏ ,

∴ ⊏ ;

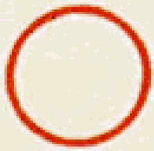
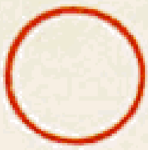
but = ,

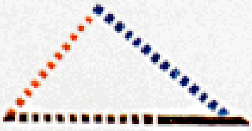
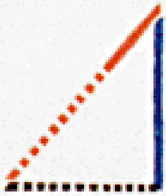
and ∴ ⊏ , a part greater than the whole, which is abſurd. Therefore the point does not fall outſide the circle, and therefore the ſtraight line cuts the circle.

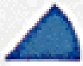
Q. E. D.



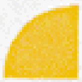
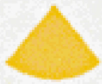
T*O draw a tangent to a given circle*  *from a given point, either in or outſide of its circumference.*

If the given point be in the circumference, as at , it is plain that the ſtraight line ⊥  the radius, will be the required tangent (B. 3. pr. 16.) But if the given point  be outſide of the circumference, draw from it to the centre, cutting ; and draw ⊥ , deſcribe  concentric with  radius = , then will be the tangent required.

For in  and 

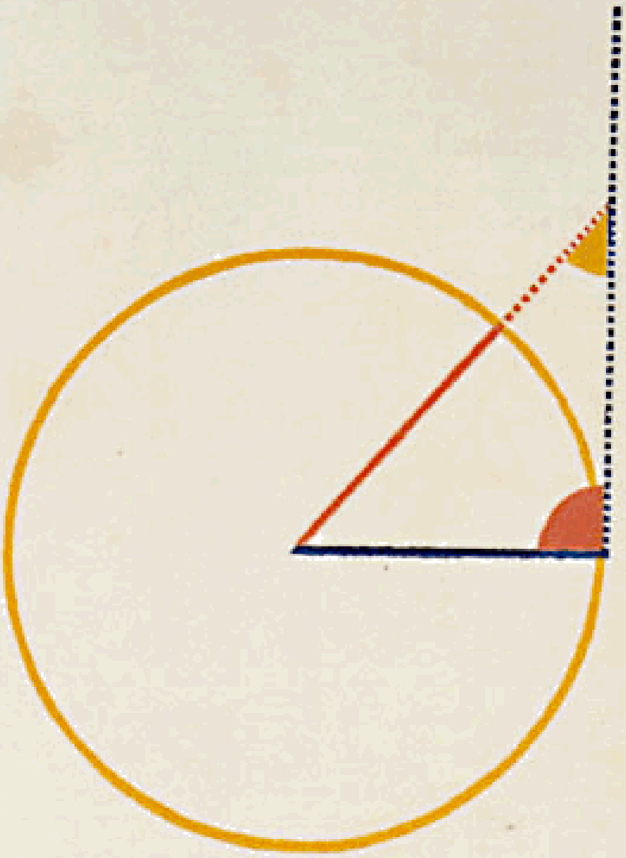
 = ,  common,

and = ,

∴ (B. 1. pr. 4.)  =  = a right angle,

∴ is a tangent to .

Q. E. D.



IF a right line be a tangent to a circle, the ſtraight line drawn from the centre to the point of contact, is perpendicular to it.

For, if it be poſſible,

let be ⊥ ,

then becauſe = ,

is acute (B. 1. pr. 17.)

∴ ⊏

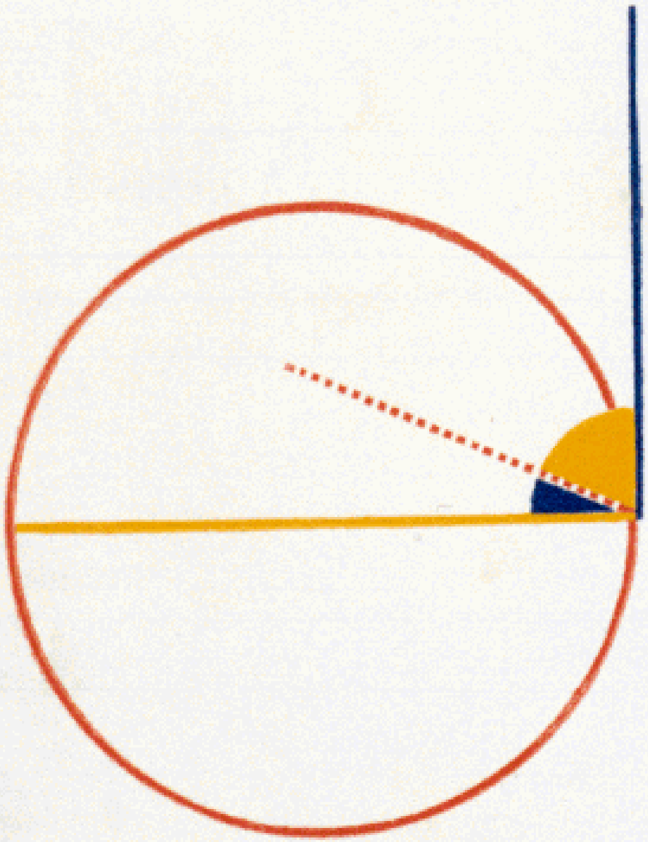
(B. 1. pr. 19.);

but = ,

and ∴ ⊏ , a part greater than the whole, which is abſurd.

∴ is not ⊥ ; and in the ſame manner it can be demonſtrated, that no other line except is perpendicular to .

Q. E. D.



IF a straight line be a tangent to a circle, the straight line , drawn perpendicular to it from point of the contact, passes through the centre of the circle.

For, if it be possible, let the centre be without , and draw from the supposed centre to the point of contact.

Because ⊥ (B. 3. pr. 18.)

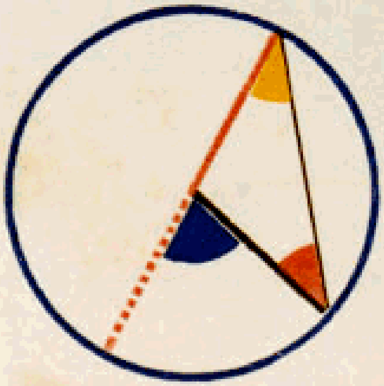
∴ = , a right angle;

but = (hyp.), and ∴ = , a part equal to the whole, which is absurd.

Therefore the assumed point is not the centre; and in the same manner it can be demonstrated, that no other point without is the centre.

Q. E. D.

FIGURE I



THE angle at the centre of a circle, is double the angle at the circumference, when they have the ſame part of the circumference for their baſe.

FIGURE I.

Let the centre of the circle be on 

a ſide of .

Becauſe  = ,

 =  (B. 1. pr. 5.).

But  =  + ,

or  = twice  (B. 1. pr. 32).

FIGURE II.

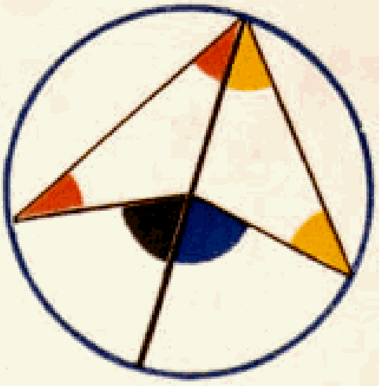
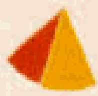


FIGURE II.

Let the centre be within , the angle at the circumference; draw  from the angular point through the centre of the circle;

then  = , and  = ,

becauſe of the equality of the ſides (B. 1. pr. 5).

Hence + + + = twice .

But = + , and

= + ,

∴ = twice .

FIGURE III.

Let the centre be without and

draw , the diameter.

Becauſe = twice ; and

= twice (caſe 1.);

∴ = twice .

Q. E. D.

FIGURE III.

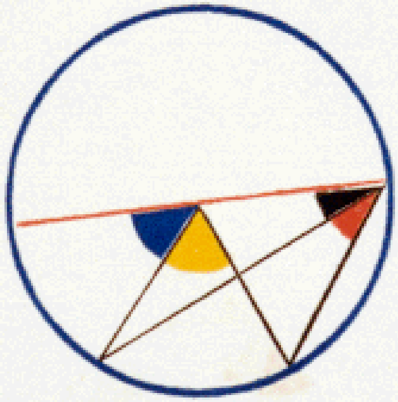
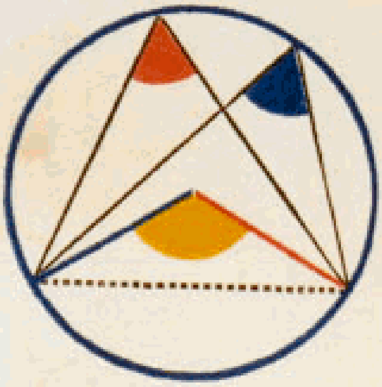


FIGURE I.



THE *angles* ( , ) *in the fame fegment of a circle are equal.*

FIGURE I.

Let the fegment be greater than a femicircle, and draw and to the centre.

 = twice  or twice = 
(B. 3. pr. 20.);

∴  = .

FIGURE II.

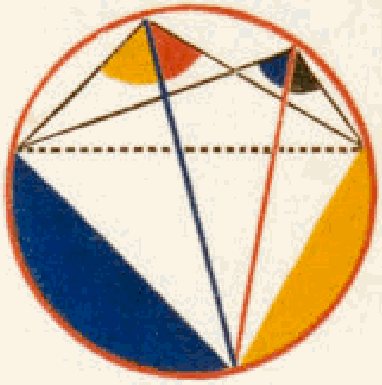
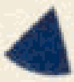
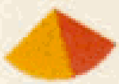
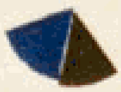


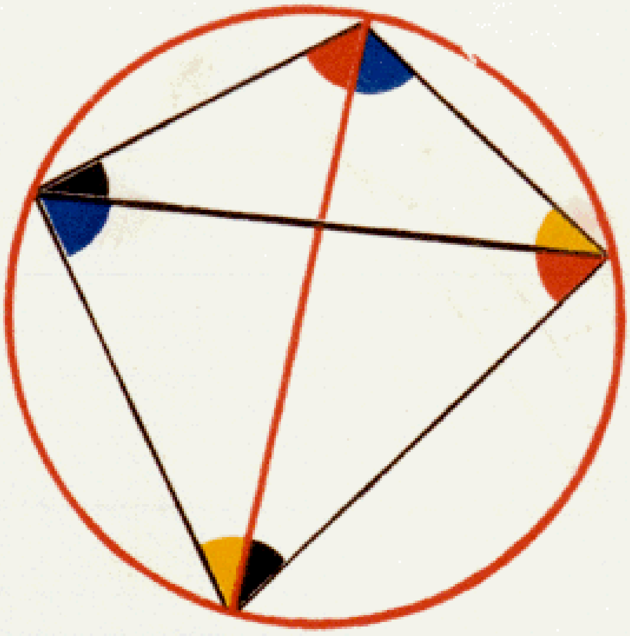
FIGURE II.

Let the fegment be a femicircle, or lefs than a femicircle, draw the diameter, alfo draw .

 =  and  =  (cafe 1.)

∴  = .

Q. E. D.



THE *oppoſite angles* and , and *of any quadrilateral figure inſcribed in a circle, are together equal to two right angles.*

Draw and the diagonals; and becauſe angles in the ſame ſegment are equal = ,

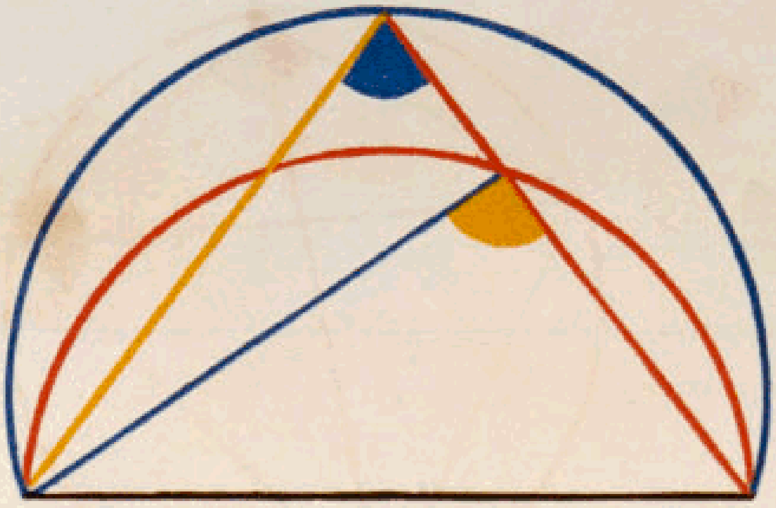
and = ;

add to both.

∴ + = + + = two right angles (B. 1. pr. 32.). In like manner it may be ſhown that,

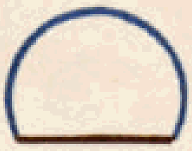
+ = .

Q. E. D.



U*PON the ſame ſtraight line, and upon the ſame ſide of it, two ſimilar ſegments of circles cannot be conſtructed which do not coincide.*

For if it be poſſible, let two ſimilar ſegments

 and  be conſtructed ;

draw any right line cutting both the ſegments,

draw and .

Becauſe the ſegments are ſimilar,

 =  (B. 3. def. 10.),

but  $\sqsubset$  (B. 1. pr. 16.)

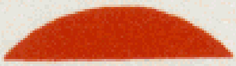
which is abſurd : therefore no point in either of the ſegments falls without the other, and therefore the ſegments coincide.

Q. E. D.

SIMILAR *segments*

 *and*

, *of circles upon equal straight lines (*  *and* *) are each equal to the other.*

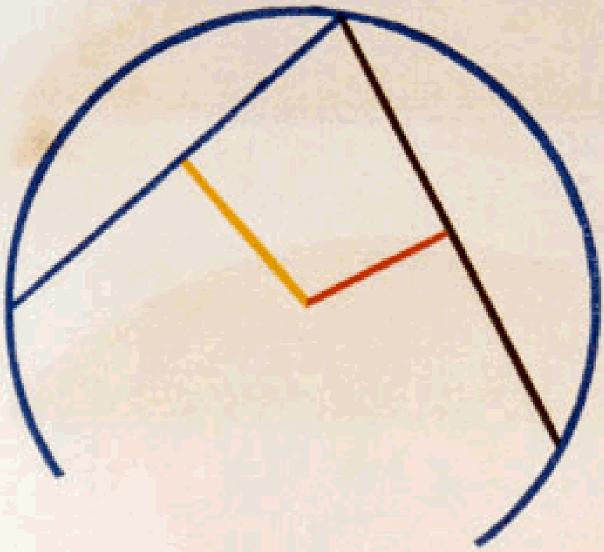
For, if  be so applied to , that may fall on , the extremities of may be on the extremities and  at the same side as ;

because = ,

 must wholly coincide with ;

and the similar segments being then upon the same straight line and at the same side of it, must also coincide (B. 3. pr. 23.), and are therefore equal.

Q. E. D.



A SEGMENT *of a circle being given, to deſcribe the circle of which it is the ſegment.*

From any point in the ſegment draw and biſect them, and from the points of biſection

draw ⊥

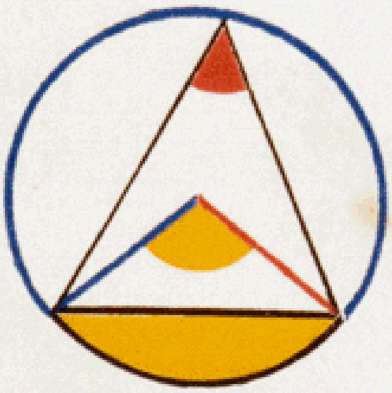
and  ⊥

where they meet is the centre of the circle.

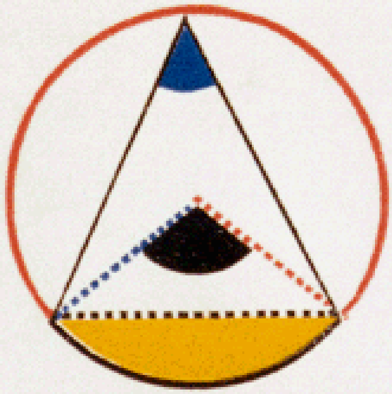
Becauſe terminated in the circle is biſected perpendicularly by , it paſſes through the centre (B. 3. pr. 1.), likewiſe paſſes through the centre, therefore the centre is in the interſection of theſe perpendiculars.

Q. E. D.

IN equal circles and , the arcs , on which ſtand equal angles, whether at the centre or circumference, are equal.



Firſt, let = at the centre, draw and .



Then ſince = ,

and have

= = = ,

and = ,

∴ = (B. 1. pr. 4.).

But = (B. 3. pr. 20.);

∴ and are ſimilar (B. 3. def. 10.);

they are alſo equal (B. 3. pr. 24.)

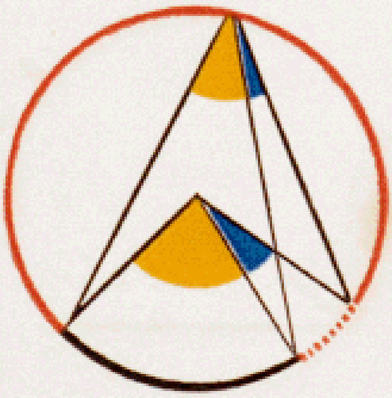
If therefore the equal ſegments be taken from the equal circles, the remaining ſegments will be equal;

hence  =  (ax. 3.);

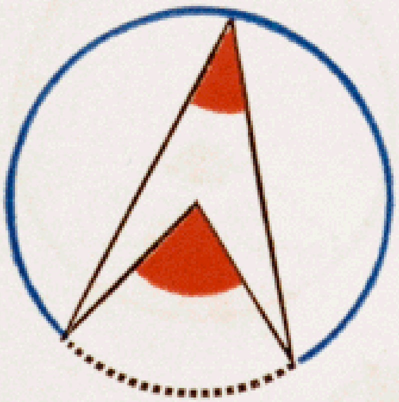
and ∴  = .

But if the given equal angles be at the circumference, it is evident that the angles at the centre, being double of thoſe at the circumference, are alſo equal, and therefore the arcs on which they ſtand are equal.

Q. E. D.



IN *equal circles,* *an* *the angles* *and* *which ſtand upon equal arches are equal, whether they be at the centres or at the circumferences.*



For if it be poſſible, let one of them

be greater than the other

and make

=

∴ = (B. 3. pr. 26.)

but = (hyp.)

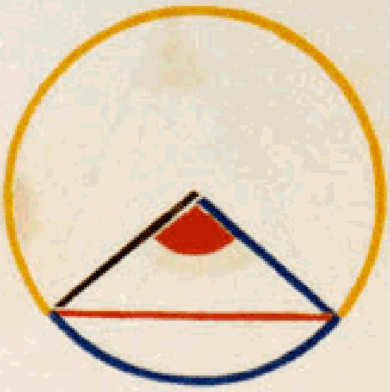
∴ = a part equal

to the whole, which is abſurd ; ∴ neither angle

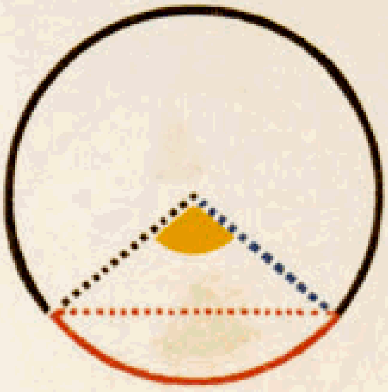
is greater than the other, and

∴ they are equal.

Q. E. D.



IN *equal circles* and , *equal chords* , *cut off equal arches.*



From the centres of the equal circles, draw , and , ;

and because =

, = ,

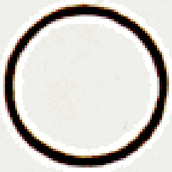
also = (hyp.)

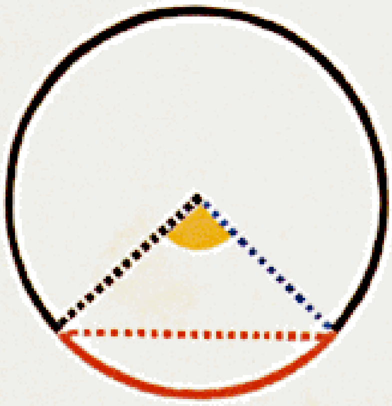
∴ =

∴ = (B. 3. pr. 26.)

and ∴ = (ax. 3.)

Q. E. D.

IN *equal circles*  *and*  *the chords* *and* *which subtend equal arcs are equal.*



If the equal arcs be ſemicircles the propoſition is evident. But if not,

let , , and , be drawn to the centres ;

becauſe  =  (hyp.)

and  =  (B. 3. pr. 27.) ;

but  and = and

∴  =  (B. 1. pr. 4.) ;

but theſe are the chords ſubtending the equal arcs.

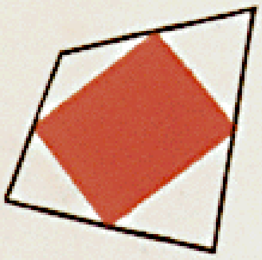
Q. E. D.

BOOK IV.

DEFINITIONS.

I.

A RECTILINEAR figure is ſaid to be *inſcribed in* another, when all the angular points of the inſcribed figure are on the ſides of the figure in which it is ſaid to be inſcribed.

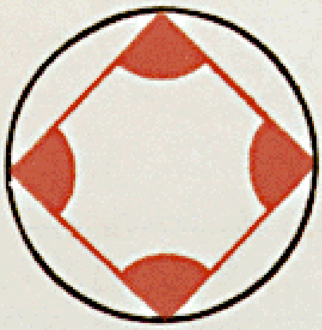


II.

A FIGURE is ſaid to be *deſcribed about* another figure, when all the ſides of the circumſcribed figure paſs through the angular points of the other figure.

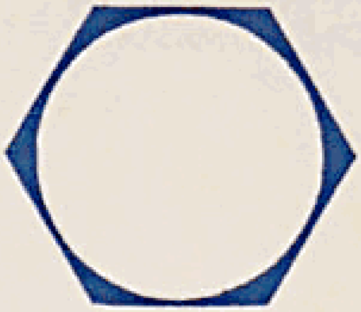
III.

A RECTILINEAR figure is ſaid to be *inſcribed in* a circle, when the vertex of each angle of the figure is in the circumference of the circle.



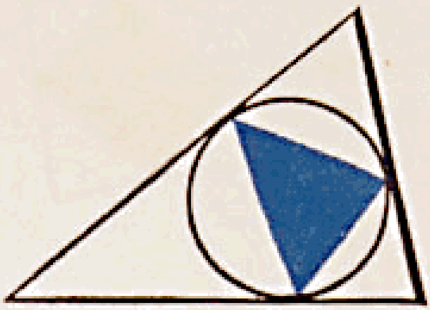
IV.

A RECTILINEAR figure is ſaid to be *circumſcribed about* a circle, when each of its ſides is a tangent to the circle.



V.

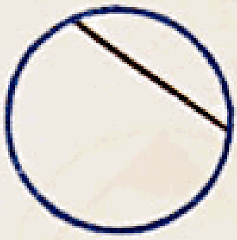
A CIRCLE is ſaid to be *inſcribed in* a rectilinear figure, when each ſide of the figure is a tangent to the circle.



VI.

A CIRCLE is ſaid to be *circumſcribed about* a rectilinear figure, when the circumference paſſes through the vertex of each angle of the figure.

 is circumſcribed.

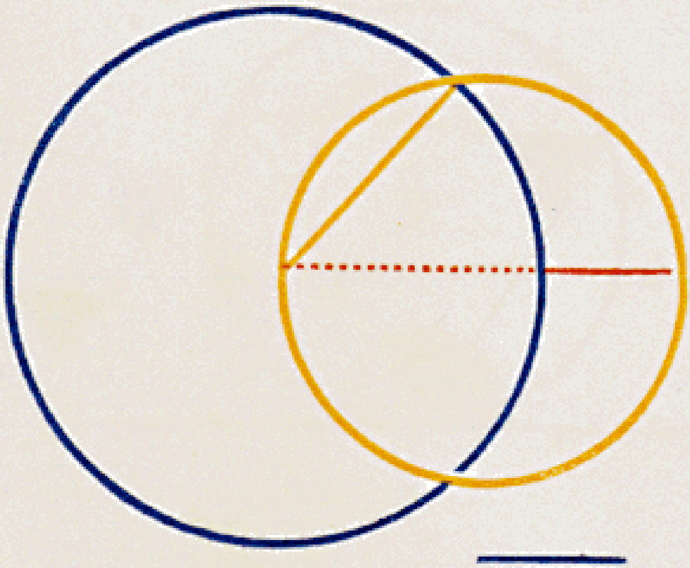


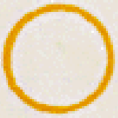
VII.

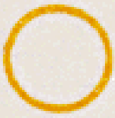
A STRAIGHT line is ſaid to be *inſcribed in* a circle, when its extremities are in the circumference.

The Fourth Book of the Elements is devoted to the ſolution of problems, chiefly relating to the inſcription and circumſcription of regular polygons and circles.

A regular polygon is one whoſe angles and ſides are equal.



IN *a given circle*  *to place a ſtraight line, equal to a given ſtraight line* ( ), *not greater than the diameter of the circle.*

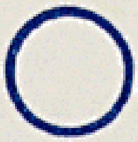
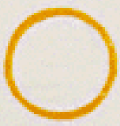
Draw , the diameter of  ;

and if = , then the problem is ſolved.

But if be not equal to ,

⊏ (hyp.);

make = (B. 1. pr. 3.) with as radius,

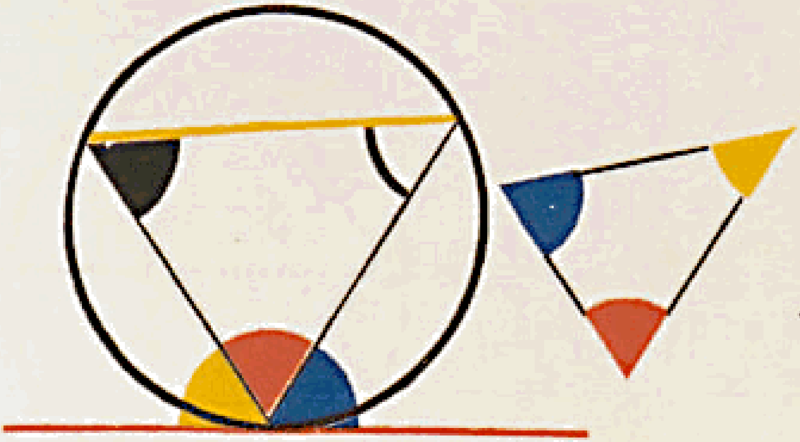
deſcribe , cutting , and

draw , which is the line required.

For = =

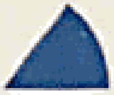
(B. 1. def. 15. conſt.)

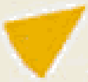
Q. E. D.

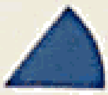


IN a given circle  *to inscribe a triangle equiangular to a given triangle.*

To any point of the given circle draw , a tangent (B. 3. pr. 17.); and at the point of contact

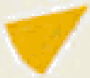
make  =  (B. 1. pr. 23.)

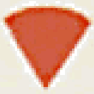
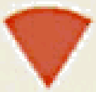
and in like manner  = , and draw .

Becaufe  =  (conft.)

and  =  (B. 3. pr. 32.)

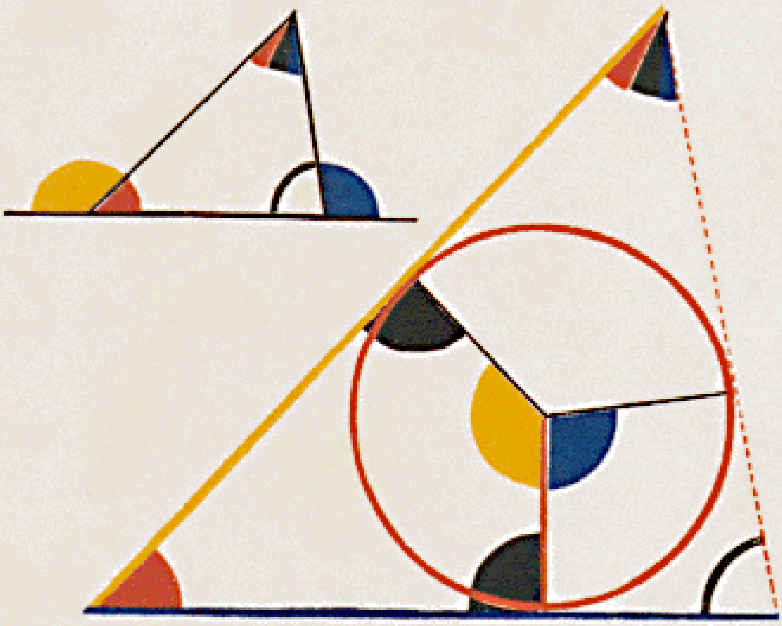
∴  = ; alfo

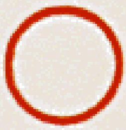
 =  for the fame reafon.

∴  =  (B. 1. pr. 32.),

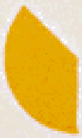
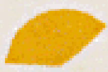
and therefore the triangle infcribed in the circle is equiangular to the given one.

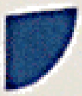
Q. E. D.



ABOUT *a given circle*  *to circumscribe a triangle equiangular to a given triangle.*

Produce any side ———, of the given triangle both ways; from the centre of the given circle draw ———, any radius.

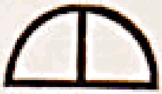
Make  =  (B. 1. pr. 23.)

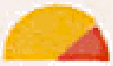
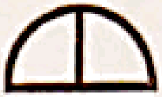
and  = .

At the extremities of the three radii, draw ———, ——— and - - - - -, tangents to the given circle. (B. 3. pr. 17.)

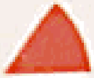
The four angles of , taken together, are equal to four right angles. (B. 1. pr. 32.)

but  and  are right angles (conſt.)

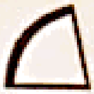
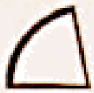
∴  +  = , two right angles

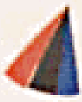
but  =  (B. 1. pr. 13.)

and  =  (conſt.)

and ∴  = .

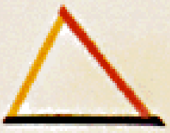
In the ſame manner it can be demonſtrated that

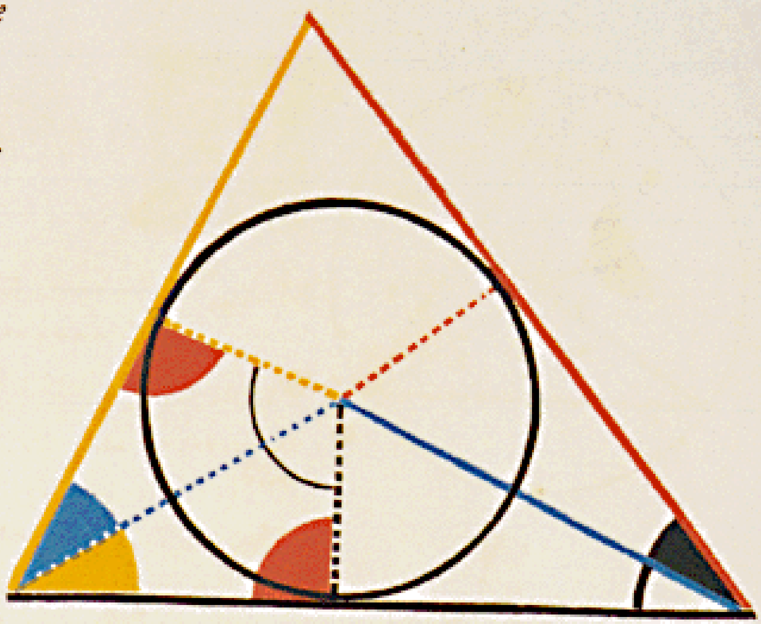
 = ;

∴  =  (B. 1. pr. 32.)

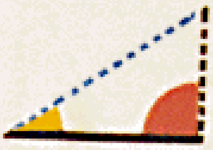
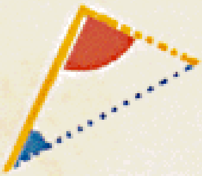
and therefore the triangle circumſcribed about the given circle is equiangular to the given triangle.

Q. E. D.

IN a given triangle  *to inscribe a circle.*



Bisect and (B. 1. pr. 9.) by and ; from the point where these lines meet draw , and respectively perpendicular to , and .

In  and 

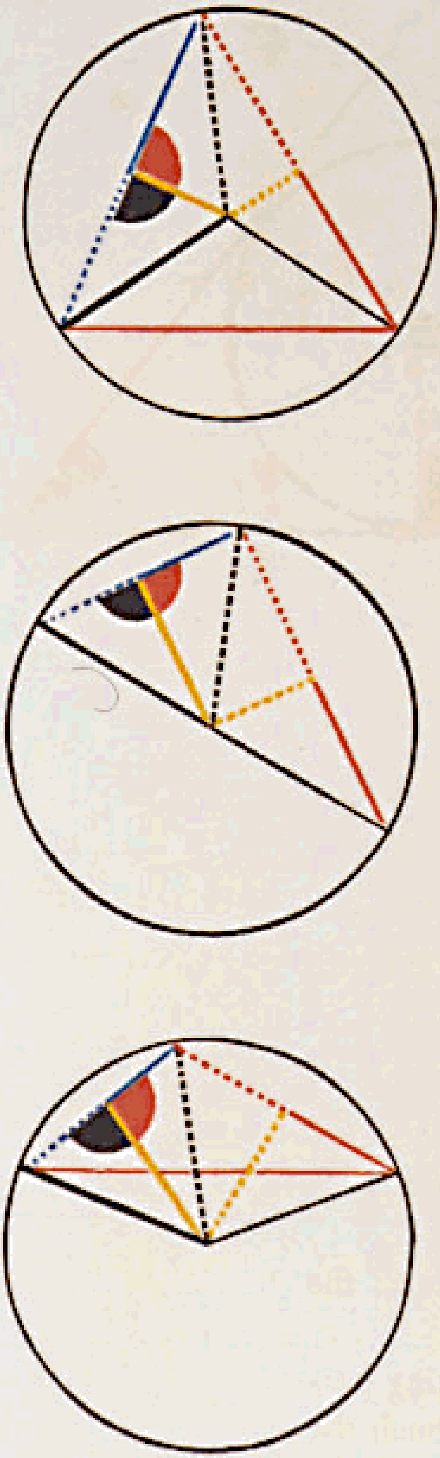
= , = and common, ∴ = (B. 1. pr. 4 and 26.)

In like manner, it may be shown also that = ,

∴ = = ; hence with any one of these lines as radius, describe

 and it will pass through the extremities of the other two; and the sides of the given triangle, being perpendicular to the three radii at their extremities, touch the circle (B. 3. pr. 16.), which is therefore inscribed in the given circle.

Q. E. D.



TO *describe a circle about a given triangle.*

Make = and = (B. 1. pr. 10.)

From the points of bisection draw and ⊥ and respectively (B. 1. pr. 11.), and from their point of concourse draw , and and describe a circle with any one of them, and it will be the circle required.

In  and 

= (const.),

common,

 =  (const.),

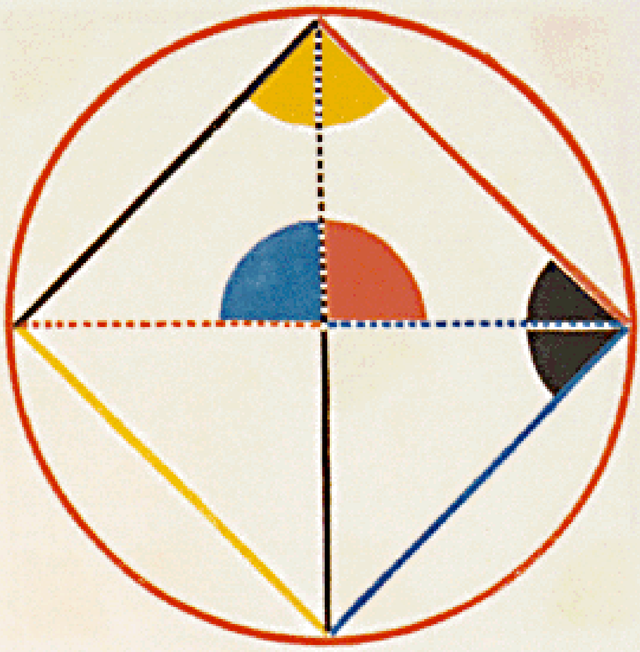
∴ = (B. 1. pr. 4.).

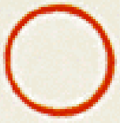
In like manner it may be shown that

= .

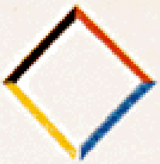
∴ = = ; and therefore a circle described from the concourse of these three lines with any one of them as a radius will circumscribe the given triangle.

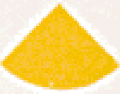
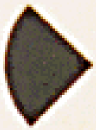
Q. E. D.



IN *a given circle*  *to inſcribe a ſquare.*

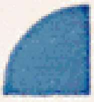
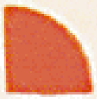
Draw the two diameters of the circle ⊥ to each other, and draw , , and

 is a ſquare.

For, ſince  and  are, each of them, in a ſemicircle, they are right angles (B. 3. pr. 31),

∴ || (B. 1. pr. 28):

and in like manner || .

And becauſe  =  (conſt.), and

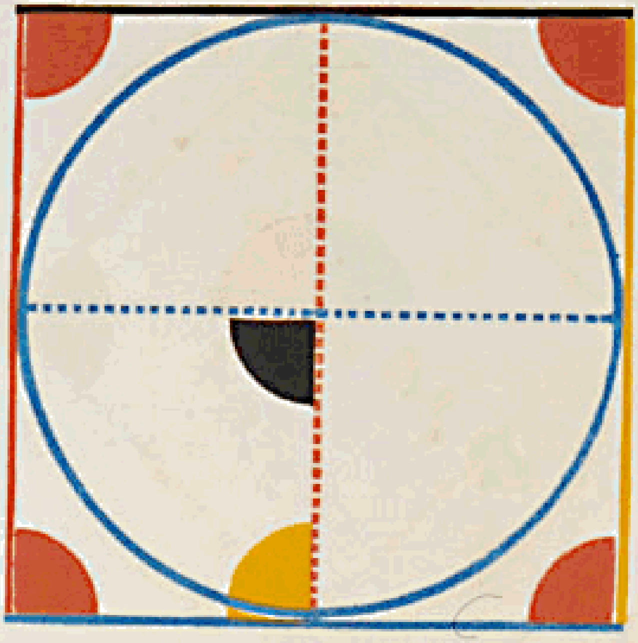
= = (B. 1. def. 15).

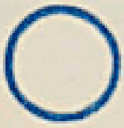
∴ = (B. 1. pr. 4);

and ſince the adjacent ſides and angles of the parallelogram  are equal, they are all equal (B. 1. pr. 34);

and ∴ , inſcribed in the given circle, is a ſquare.

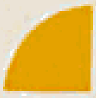
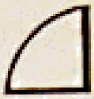
Q. E. D.

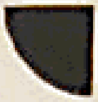
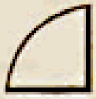


ABOUT *a given circle*  *to circumscribe a square.*

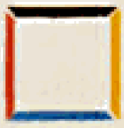
Draw two diameters of the given circle perpendicular to each other, and through their extremities draw , , , and tangents to the circle;

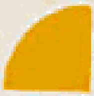
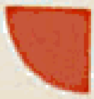
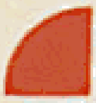
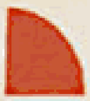
and  is a square.

 =  a right angle, (B. 3. pr. 18.)

also  =  (const.),

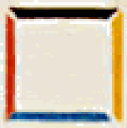
∴ || ; in the same manner it can be demonstrated that || , and also that and || ;

∴  is a parallelogram, and

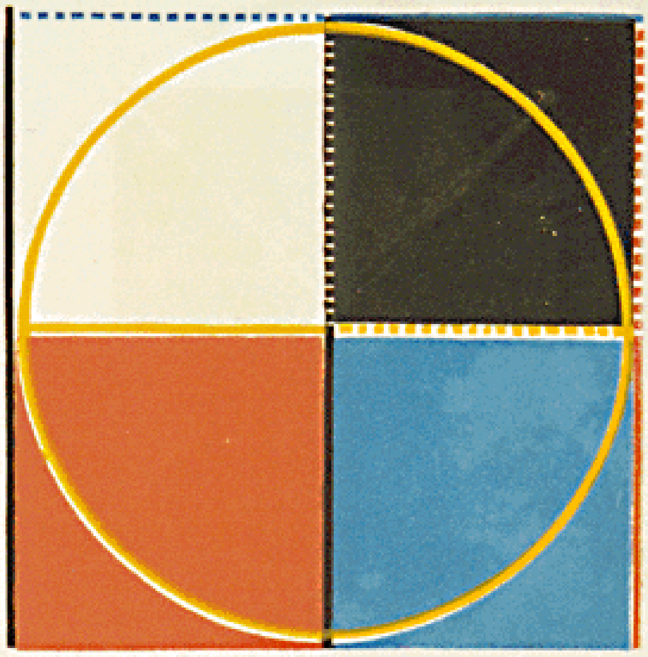
because  =  =  =  = 

they are all right angles (B. 1. pr. 34):

it is also evident that , , and are equal.

∴  is a square.

Q. E. D.



TO inscribe a circle in a given square.

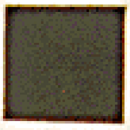
Make = ,

and = ,

draw ∥ ,

and ∥

(B. 1. pr. 31.)

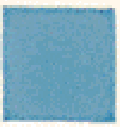
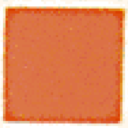
∴  is a parallelogram;

and since = (hyp.)

=

∴  is equilateral (B. 1. pr. 34.)

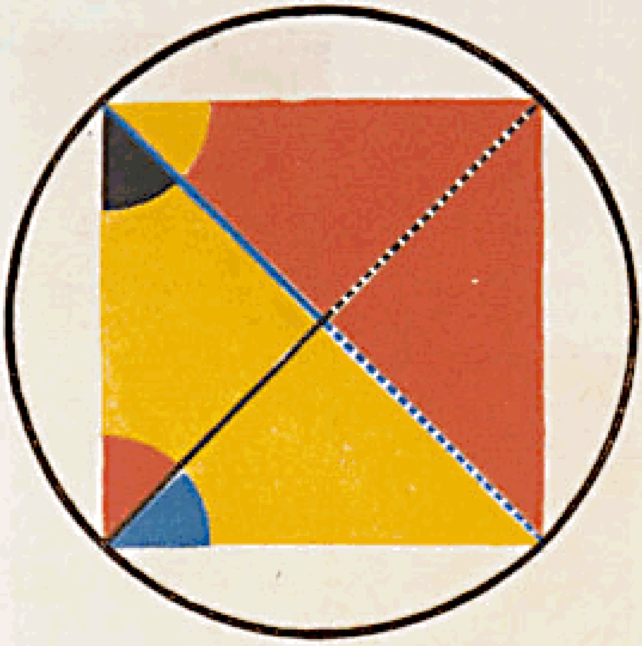
In like manner, it can be shown that

 =  are equilateral parallelograms;

∴ = = = ,

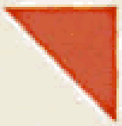
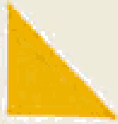
and therefore if a circle be described from the concourse of these lines with any one of them as radius, it will be inscribed in the given square. (B. 3. pr. 16.)

Q. E. D.



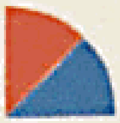
TO *describe a circle about a given square*  .

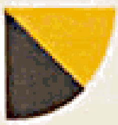
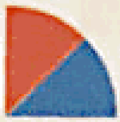
Draw the diagonals ——— and ——— intersecting each other; then,

because  and  have their sides equal, and the base ——— common to both,

 =  (B. 1. pr. 8),

or  is bisected: in like manner it can be shown

that  is bisected;

but  = ,

hence  =  their halves,

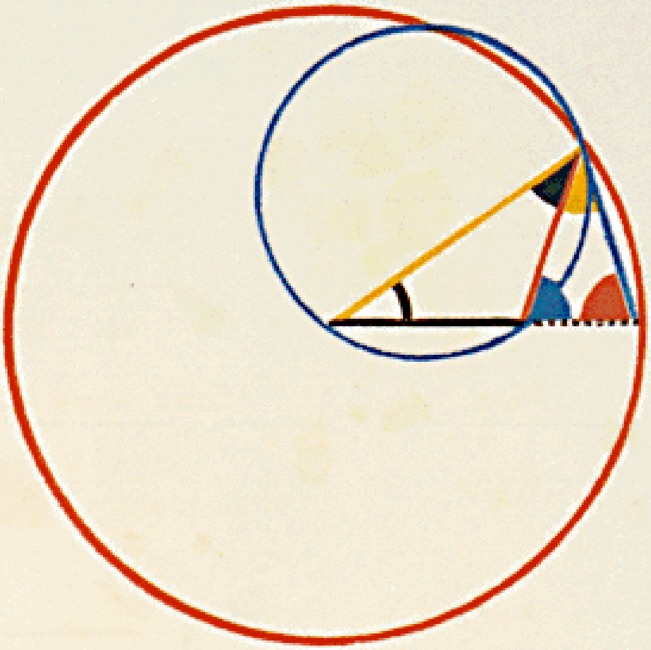
∴ ——— = ———; (B. 1. pr. 6.)

and in like manner it can be proved that

——— = ——— = ——— = ———.

If from the confluence of these lines with any one of them as radius, a circle be described, it will circumscribe the given square.

Q. E. D.



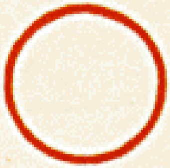
To construct an isosceles triangle, in which each of the angles at the base shall be double of the vertical angle.

Take any straight line ——

and divide it so that

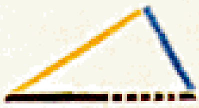
—— × —— = ——²

(B. 2. pr. 11.)

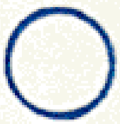
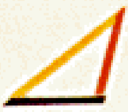
With —— as radius, describe  and place

in it from the extremity of the radius, —— = ——,

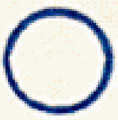
(B. 4. pr. 1); draw ——.

Then  is the required triangle.

For, draw —— and describe

 about  (B. 4. pr. 5.)

Since —— × —— = ——² = ——²,

∴ —— is a tangent to  (B. 3. pr. 37.)

∴  =  (B. 3. pr. 32), add to each,

∴ + = + ;

but + or = (B. 1. pr. 5):

since = (B. 1. pr. 5.)

consequently = + = (B. 1. pr. 32.)

∴ = (B. 1. pr. 6.)

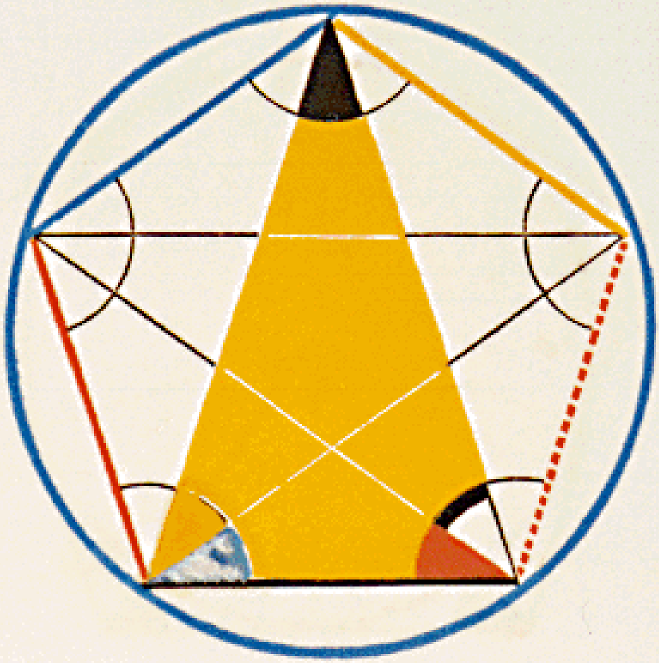
∴ = = (const.)

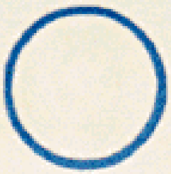
∴ = (B. 1. pr. 5.)

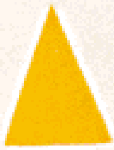
∴ = = = +

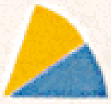
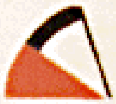
= twice ; and consequently each angle at the base is double of the vertical angle.

Q. E. D.



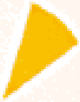
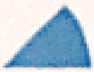
IN *a given circle*  *to inſcribe an equilateral and equiangular pentagon.*

Conſtruct an iſoſceles triangle, in which each of the angles at the baſe ſhall be double of the angle at the vertex, and inſcribe in the given circle a triangle  equiangular to it; (B. 4. pr. 2.)

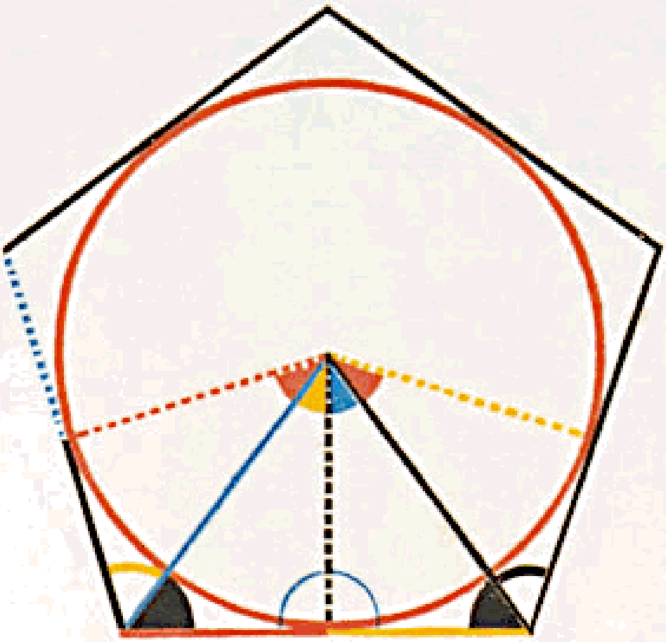
Biſect  and  (B. 1. pr. 9.)

draw ———, ———, ——— and - - - - - .

Becauſe each of the angles

, , ,  and  are equal, the arcs upon which they ſtand are equal, (B. 3. pr. 26.) and ∴ ———, ———, ———, ——— and - - - - - which ſubtend theſe arcs are equal (B. 3. pr. 29.) and ∴ the pentagon is equilateral, it is alſo equiangular, as each of its angles ſtand upon equal arcs. (B. 3. pr. 27).

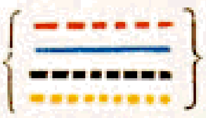
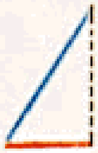
Q. E. D.



TO describe an equilateral and equiangular pentagon about a given circle .

Draw five tangents through the vertices of the angles of any regular pentagon inſcribed in the given circle  (B. 3. pr. 17).

Theſe five tangents will form the required pentagon.

Draw . In  and 

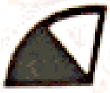
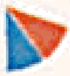
 = (B. 1. pr. 47),

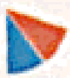
 = , and common;

∴  =  and  =  (B. 1. pr. 8.)

∴  = twice , and  = twice :

In the ſame manner it can be demonſtrated that

 = twice , and  = twice :

but  =  (B. 3. pr. 27),

∴ their halves = , alſo = , and

common;

∴ = and = ,

∴ = twice ;

In the ſame manner it can be demonſtrated

that = twice ,

but =

∴ = ;

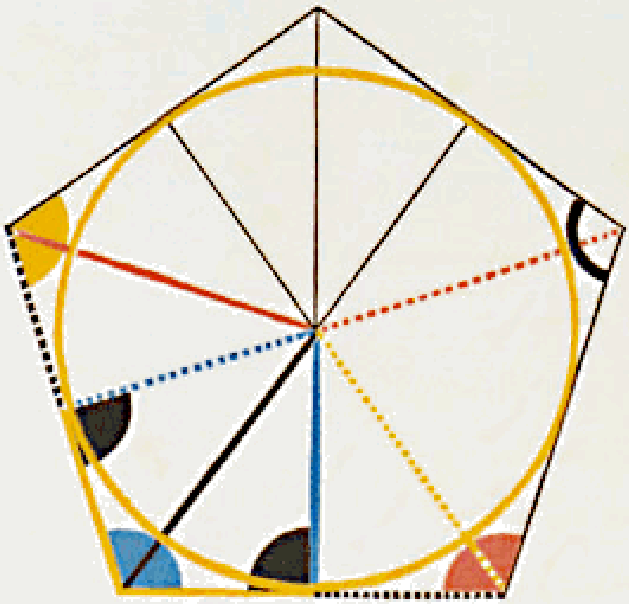
In the ſame manner it can be demonſtrated that the other ſides are equal, and therefore the pentagon is equilateral, it is alſo equiangular, for

= twice and = twice ,

and therefore = ,

∴ = ; in the ſame manner it can be demonſtrated that the other angles of the deſcribed pentagon are equal.

Q. E. D



TO *inſcribe a circle in a given equiangular and equilateral pentagon.*

Let  be a given equiangular and equilateral pentagon; it is required to inſcribe a circle in it.

Make … = …, and … = … (B. 1. pr. 9.)

Draw …, …, …, …, &c.

Becauſe … = …, … = …, and … common to the two triangles

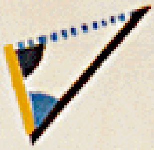
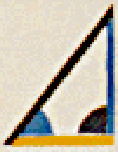
∴ … = … and … = … (B. 1. pr. 4.)

And becauſe … = … = twice …

∴ … = twice …, hence … is biſected by ….

In like manner it may be demonſtrated that … is biſected by …, and that the remaining angle of the polygon is biſected in a ſimilar manner.

Draw , , &c. perpendicular to the ſides of the pentagon.

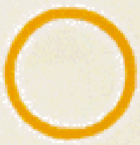
Then in the two triangles  and 

we have  = , (conſt.), common,

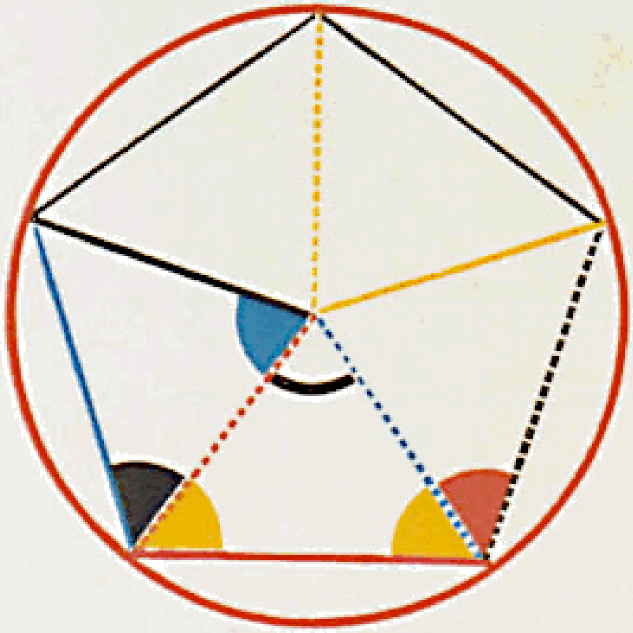
and  =  = a right angle;

∴ = . (B. 1. pr. 26.)

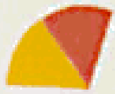
In the ſame way it may be ſhown that the five perpendiculars on the ſides of the pentagon are equal to one another.

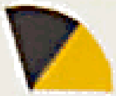
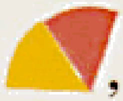
Deſcribe  with any one of the perpendiculars as radius, and it will be the inſcribed circle required. For if it does not touch the ſides of the pentagon, but cut them, then a line drawn from the extremity at right angles to the diameter of a circle will fall within the circle, which has been ſhown to be abſurd. (B. 3. pr. 16.)

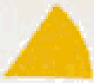
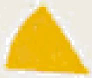
Q. E. D.

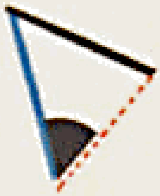
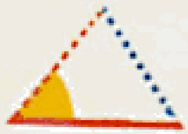


To describe a circle about a given equilateral and equiangular pentagon.

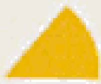
Bisect  and 
by and , and
from the point of section, draw
, , and .

 = ,

 = , ∴ = (B. 1. pr. 6);

and since in  and ,

= , and common,

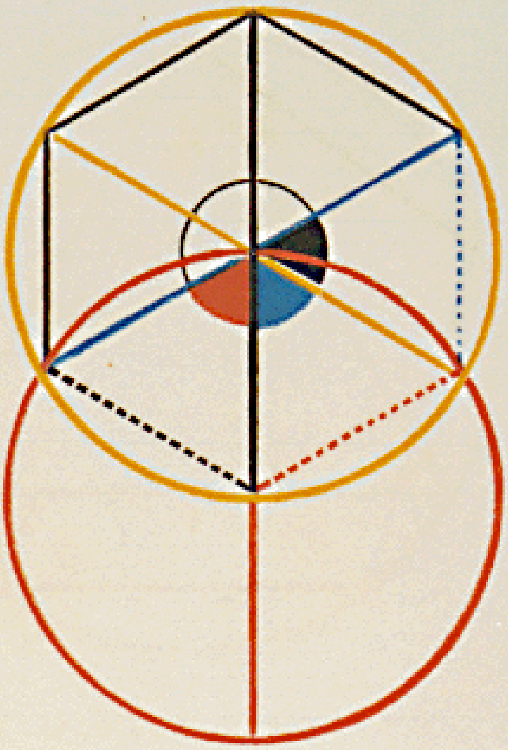
also  = ;

∴ = (B. 1. pr. 4).

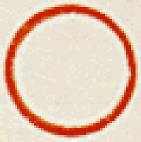
In like manner it may be proved that
= = , and
therefore = = =
= :

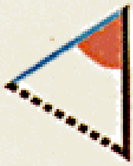
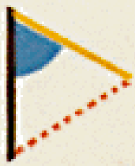
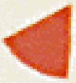
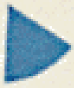
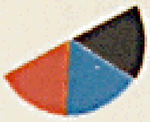
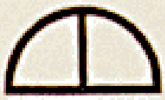
Therefore if a circle be described from the point where these five lines meet, with any one of them as a radius, it will circumscribe the given pentagon.

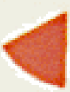
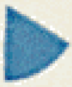
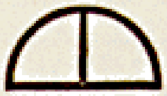
Q. E. D.



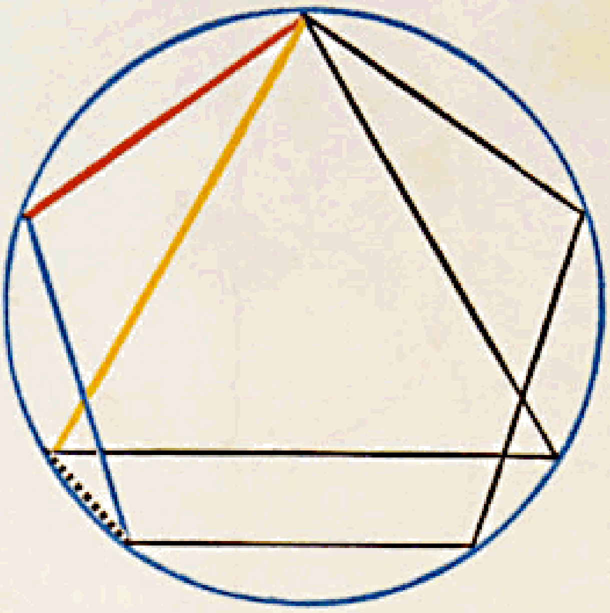
TO *inscribe an equilateral and equiangular hexagon in a given circle* .

From any point in the circumference of the given circle describe  passing through its centre, and draw the diameters , and ; draw , , , &c. and the required hexagon is inscribed in the given circle.

Since passes through the centres of the circles,  and  are equilateral triangles, hence  =  = one-third of two right angles; (B. 1. pr. 32) but  =  (B. 1. pr. 13);

∴  =  =  = one-third of  (B. 1. pr. 32), and the angles vertically opposite to these are all equal to one another (B. 1. pr. 15), and stand on equal arches (B. 3. pr. 26), which are subtended by equal chords (B. 3. pr. 29); and since each of the angles of the hexagon is double of the angle of an equilateral triangle, it is also equiangular.

Q. E. D.



TO inſcribe an equilateral and equiangular quindecagon in a given circle.

Let and be the ſides of an equilateral pentagon inſcribed in the given circle, and the ſide of an inscribed equilateral triangle.

The arc ſubtended by and $\Big\} = \frac{2}{5} = \frac{6}{15} \Big\{$ of the whole circumference.

The arc ſubtended by $\Big\} = \frac{1}{3} = \frac{5}{15} \Big\{$ of the whole circumference.

Their difference $= \frac{1}{15}$

∴ the arc ſubtended by $= \frac{1}{15}$ difference of the whole circumference.

Hence if ſtraight lines equal to be placed in the circle (B. 4. pr. 1), an equilateral and equiangular quindecagon will be thus inſcribed in the circle.

Q. E. D.

BOOK V.

DEFINITIONS.

I.

A LESS magnitude is ſaid to be an aliquot part or ſubmultiple of a greater magnitude, when the leſs meaſures the greater; that is, when the leſs is contained a certain number of times exactly in the greater.

II.

A GREATER magnitude is ſaid to be a multiple of a leſs, when the greater is meaſured by the leſs; that is, when the greater contains the leſs a certain number of times exactly.

III.

RATIO is the relation which one quantity bears to another of the ſame kind, with reſpect to magnitude.

IV.

MAGNITUDES are ſaid to have a ratio to one another, when they are of the ſame kind; and the one which is not the greater can be multiplied ſo as to exceed the other.

The other definitions will be given throughout the book where their aid is firſt required.

AXIOMS.

I.

EQUIMULTIPLES or equiſubmultiples of the ſame, or of equal magnitudes, are equal.

If A = B, then
twice A = twice B, that is,
2 A = 2 B;
3 A = 3 B;
4 A = 4 B;
&c. &c.
and $\frac{1}{2}$ of A = $\frac{1}{2}$ of B;
$\frac{1}{3}$ of A = $\frac{1}{3}$ of B;
&c. &c.

II.

A MULTIPLE of a greater magnitude is greater than the ſame multiple of a leſs.

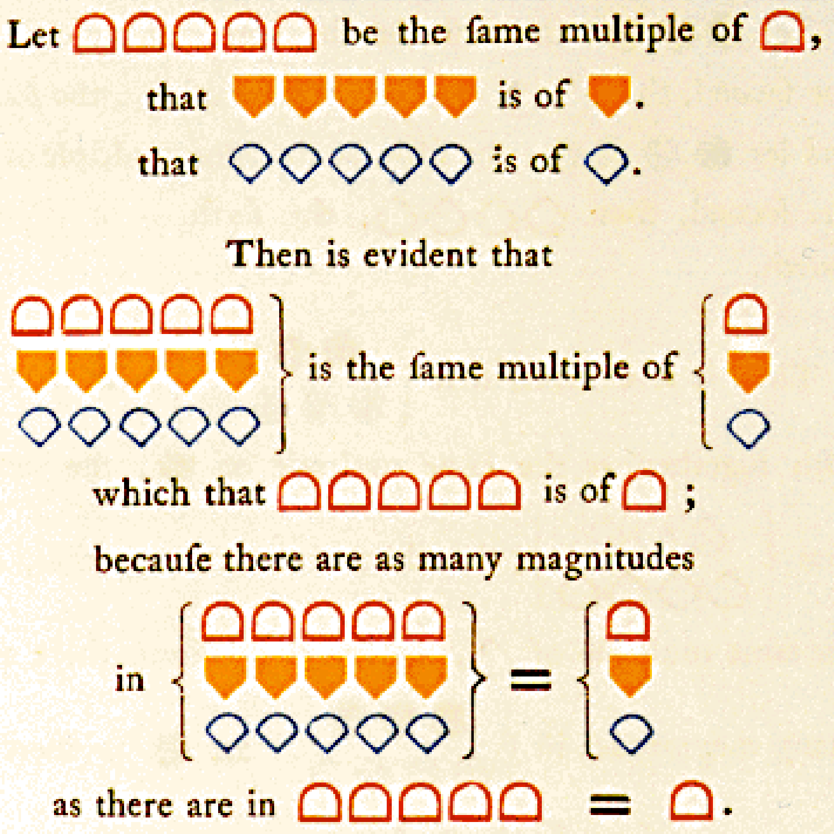
Let A ⊏ B, then
2 A ⊏ 2 B;
3 A ⊏ 3 B;
4 A ⊏ 4 B;
&c. &c.

III.

THAT magnitude, of which a multiple is greater than the ſame multiple of another, is greater than the other.

Let 2 A ⊏ 2 B, then
A ⊏ B;
or, let 3 A ⊏ 3 B, then
A ⊏ B;
or, let *m* A ⊏ *m* B, then
A ⊏ B.

IF any number of magnitudes be equimultiples of as many others, each of each: what multiple soever any one of the firſt is of its part, the ſame multiple ſhall of the firſt magnitudes taken together be of all the others taken together.

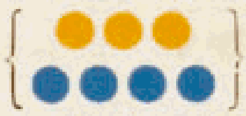
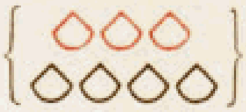
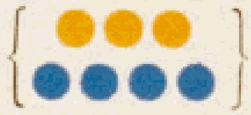
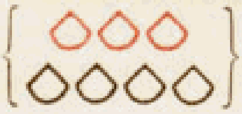


The ſame demonſtration holds in any number of magnitudes, which has here been applied to three.

∴ If any number of magnitudes, &c.

IF the firſt magnitude be the ſame multiple of the ſecond that the third is of the fourth, and the fifth the ſame multiple of the ſecond that the ſixth is of the fourth, then ſhall the firſt, together with the fifth, be the ſame multiple of the ſecond that the third, together with the ſixth, is of the fourth.

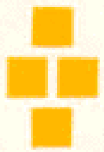
Let , the firſt, be the ſame multiple of , the ſecond, that , the third, is of , the fourth; and let , the fifth, be the ſame multiple of , the ſecond, that , the ſixth, is of , the fourth.

Then it is evident, that , the firſt and fifth together, is the ſame multiple of , the ſecond, that , the third and ſixth together, is of the ſame multiple of , the fourth; becauſe there are as many magnitudes in  =  as there are in  = .

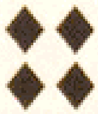
∴ If the firſt magnitude, &c.

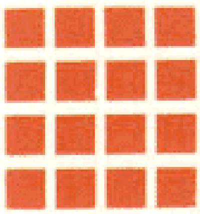
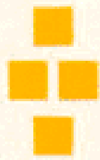
IF *the firſt of four magnitudes be the ſame multiple of the ſecond that the third is of the fourth, and if any equimultiples whatever of the firſt and third be taken, thoſe ſhall be equimultiples; one of the ſecond, and the other of the fourth.*

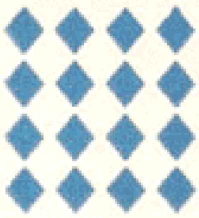
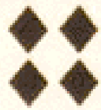
The First. The Second.

Let  be the ſame multiple of 

The Third. The Fourth.

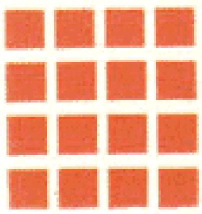
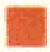
which  is of ;

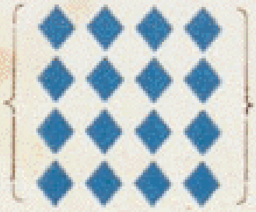
take  the ſame multiple of ,

which  is of .

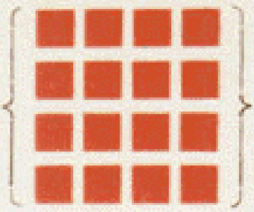
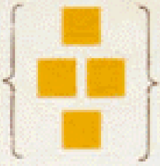
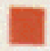
Then it is evident,

The Second.

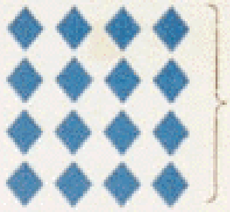
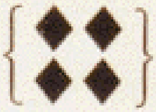
that  is the ſame multiple of 

which  is of  ;

The Fourth.

because  contains  contains 

as many times as

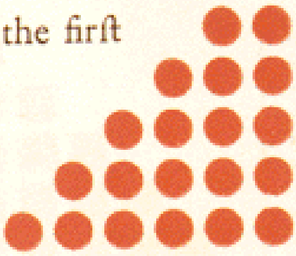
 contains  contains  .

The ſame reaſoning is applicable in all caſes.

∴ If the firſt four, &c.

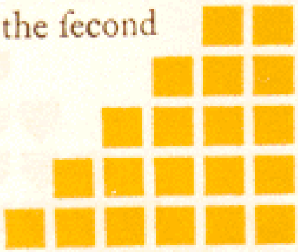
DEFINITION V.

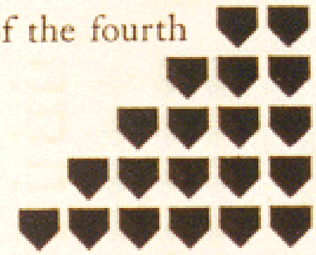
Four magnitudes, , , , , are said to be proportionals when every equimultiple of the first and third be taken, and every equimultiple of the second and fourth, as,

of the first

&c.

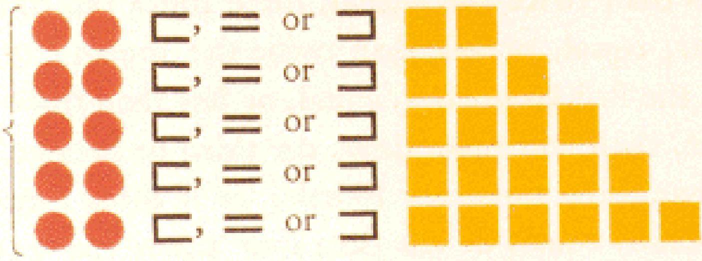
of the third

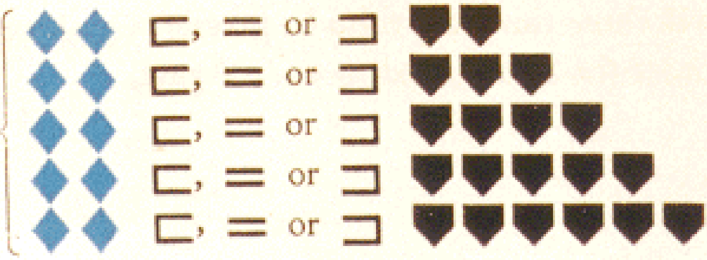
&c.

of the second

&c.

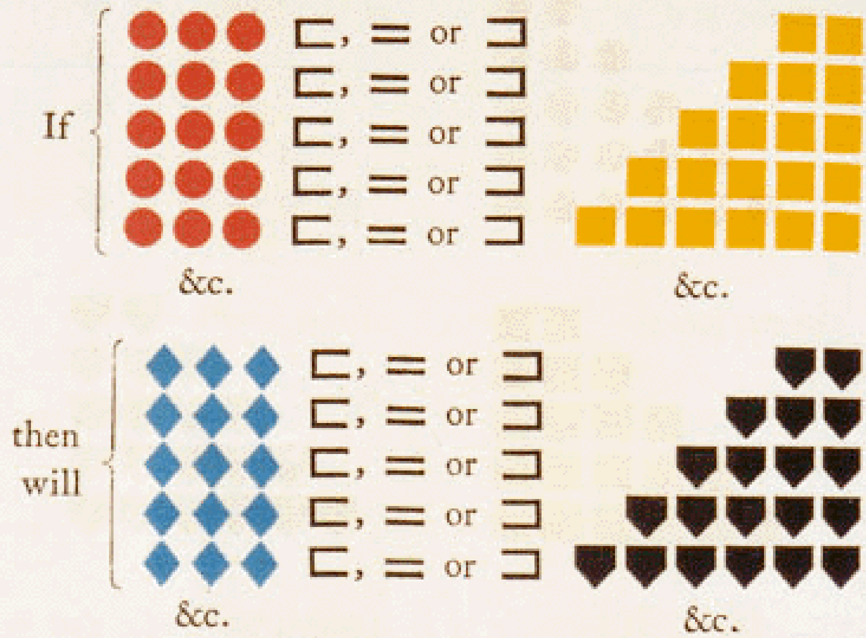
of the fourth

&c.

Then taking every pair of equimultiples of the first and third, and every pair of equimultiples of the second and fourth,

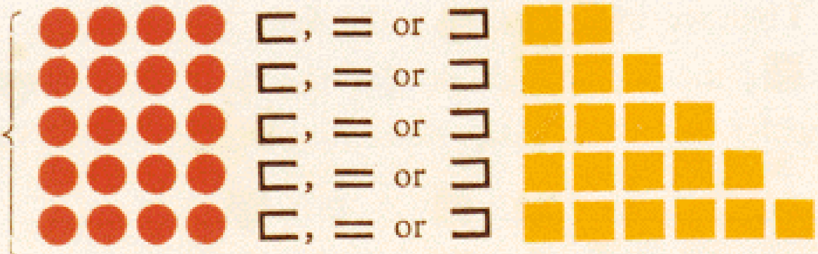
If


then will


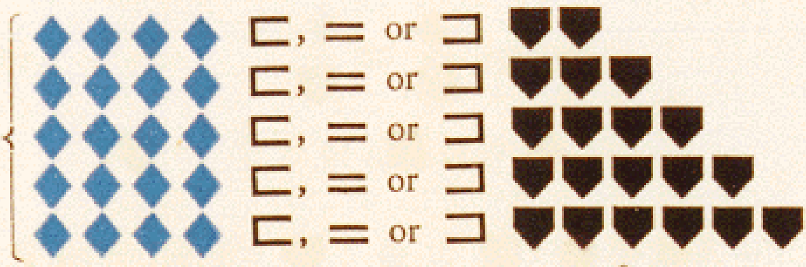
That is, if twice the firſt be greater, equal, or leſs than twice the ſecond, twice the third will be greater, equal, or leſs than twice the fourth; or, if twice the firſt be greater, equal, or leſs than three times the ſecond, twice the third will be greater, equal, or leſs than three times the fourth, and so on, as above expreſſed.



In other terms, if three times the firſt be greater, equal, or leſs than twice the ſecond, three times the third will be greater, equal, or leſs than twice the fourth; or, if three times the firſt be greater, equal, or leſs than three times the ſecond, then will three times the third be greater, equal, or leſs than three times the fourth; or if three times the firſt be greater, equal, or leſs than four times the ſecond, then will three times the third be greater, equal, or leſs than four times the fourth, and so on. Again,

If


&c. &c.

then will


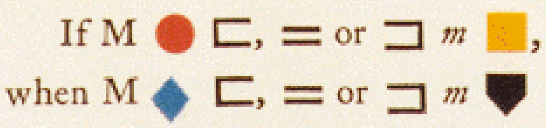
&c. &c.

And so on, with any other equimultiples of the four magnitudes, taken in the ſame manner.

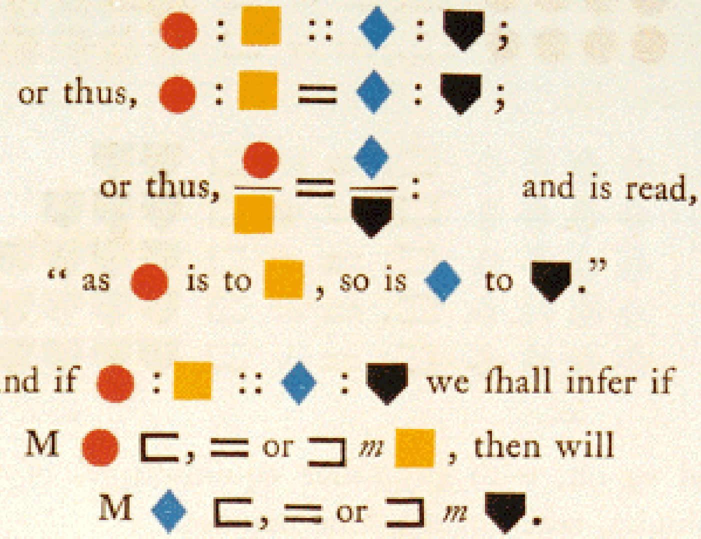
Euclid expreſſes this definition as follows:—

The firſt of four magnitudes is ſaid to have the ſame ratio to the ſecond, which the third has to the fourth, when any equimultiples whatſoever of the firſt and third being taken, and any equimultiples whatſoever of the ſecond and fourth; if the multiple of the firſt be leſs than that of the ſecond, the multiple of the third is alſo leſs than that of the fourth; or, if the multiple of the firſt be equal to that of the ſecond, the multiple of the third is alſo equal to that of the fourth; or, if the multiple of the firſt be greater than that of the ſecond, the multiple of the third is alſo greater than that of the fourth.

In future we ſhall expreſs this definition generally, thus:



Then we infer that , the firſt, has the ſame ratio to , the ſecond, which , the third, has to  the fourth: expreſſed in the ſucceeding demonſtrations thus:



That is, if the firſt be to the ſecond, as the third is to the fourth; then if M times the firſt be greater than, equal to, or leſs than *m* times the ſecond, then ſhall M times the third be greater than, equal to, or leſs than *m* times the fourth, in which M and *m* are not to be conſidered particular multiples, but every pair of multiples whatever; nor are ſuch marks as , , , &c. to be conſidered any more than repreſentatives of geometrical magnitudes.

The ſtudent ſhould thoroughly underſtand this definition before proceeding further.

IF the firſt of four magnitudes have the ſame ratio to the ſecond, which the third has to the fourth, then any equimultiples whatever of the firſt and third shall have the ſame ratio to any equimultiples of the ſecond and fourth; viz., the equimultiple of the firſt ſhall have the ſame ratio to that of the ſecond, which the equimultiple of the third has to that of the fourth.

Let ● : ■ :: ◆ : ⬟, then 3 ● : 2 ■ :: 3 ◆ : 2 ⬟, every equimultiple of 3 ● and 3 ◆ are equimultiples of ● and ◆, and every equimultiple of 2 ■ and 2 ⬟, are equimultiples of ■ and ⬟ (B. 5, pr. 3.)

That is, M times 3 ● and M times 3 ◆ are equimultiples of ● and ◆, and m times 2 ■ and m 2 ⬟ are equimultiples of 2 ■ and 2 ⬟; but ● : ■ :: ◆ : ⬟ (hyp); ∴ if M 3 ● $\sqsubset$, $=$, or $\sqsupset$ m 2 ■, then

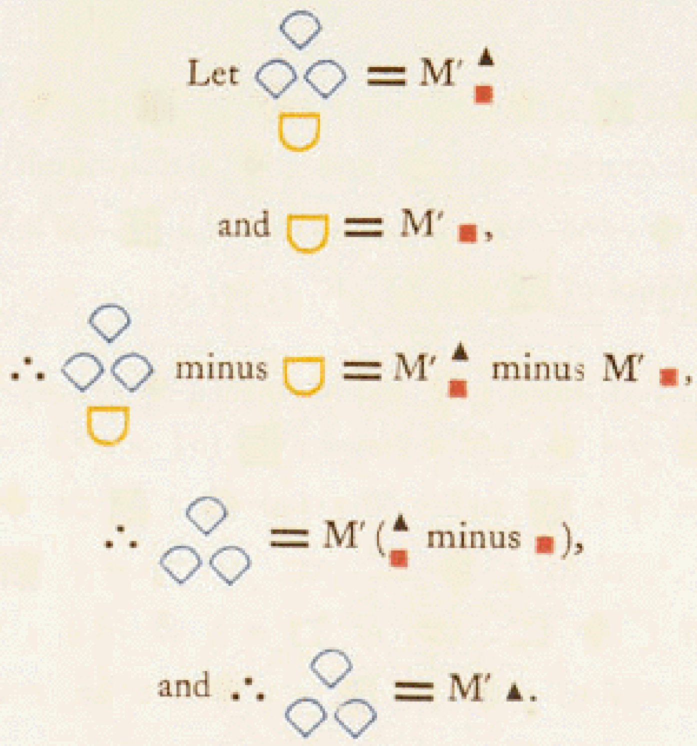
M 3 ◆ $\sqsubset$, $=$, or $\sqsupset$ m 2 ⬟ (def. 5.)

and therefore 3 ● : 2 ■ :: 3 ◆ : 2 ⬟ (def. 5.)

The ſame reaſoning holds good if any other equimultiple of the firſt and third be taken, any other equimultiple of the ſecond and fourth.

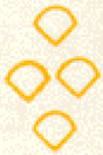
∴ If the firſt four magnitudes, &c.

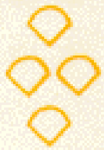
IF one magnitude be the ſame multiple of another, which a magnitude taken from the firſt is of a magnitude taken from the other, the remainder ſhall be the ſame multiple of the remainder, that the whole is of the whole.



∴ If one magnitude, &c.

IF *two magnitudes be equimultiples of two others, and if equimultiples of theſe be taken from the firſt two, the remainders are either equal to theſe others, or equimultiples of them.*

Let  = M′ ■ ; and  = M′ ▲ ;

then  minus m' ■ =

M′ ■ minus m' ■ = (M′ minus m') ■,

and  minus m' ▲ = M′ ▲ minus m' ▲ = (M′ minus m') ▲ .

Hence, (M′ minus m') ■ and (M′ minus m') ▲ are equimultiples of ■ and ▲ , and equal to ■ and ▲, when M′ minus m' = 1.

∴ If two magnitudes be equimultiples, &c.

IF the firſt of the four magnitudes has the ſame ratio to the ſecond which the third has to the fourth, then if the firſt be greater than the ſecond, the third is alſo greater than the fourth; and if equal, equal; if leſs, leſs.

Let ● : ■ :: ▼ : ◆ ; therefore, by the fifth definition, if ●● ⊏ ■■, then will ▼▼ ⊏ ◆◆ ;
but if ● ⊏ ■, then ●● ⊏ ■■
and ▼▼ ⊏ ◆◆ ,
and ∴ ▼ ⊏ ◆ .

Similarly, if ● =, or ⊐ ■, then will ▼ =, or ⊐ ◆ .

∴ If the firſt of four, &c.

DEFINITION XIV.

Geometricians make uſe of the technical term "Invertendo," by inverſion, when there are four proportionals, and it is inferred, that the ſecond is to the firſt as the fourth to the third.

Let A : B :: C : D , then, by "invertendo" it is inferred B : A :: D : C.

I*F four magnitudes are proportionals, they are proportionals also when taken inversely.*

Let ⬟ : ⬠ :: ■ : ◆ ,

then, inversely, ⬠ : ⬟ :: ◆ : ■.

If M ⬟ ⊐ *m* ⬠, then M ■ ⊐ *m* ◆

by the fifth definition.

Let M ⬟ ⊐ *m* ⬠, that is, *m* ⬠ ⊏ M ⬟,

∴ M ■ ⊐ *m* ◆ , or, *m* ◆ ⊏ M ■ ;

∴ if *m* ⬠ ⊏ M ⬟, then will *m* ◆ ⊏ M ■.

In the same manner it may be shown,

that if *m* ⬠ = or ⊐ M ⬟,

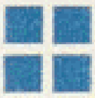
then will *m* ◆ =, or ⊐ M ■ ;

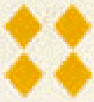
and therefore, by the fifth definition, we infer

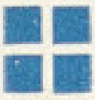
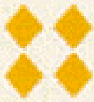
that ⬠ : ⬟ : ◆ : ■.

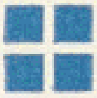
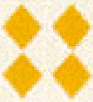
∴ If four magnitudes, &c.

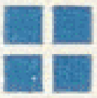
IF the first be the same multiple of the second, or the same part of it, that the third is of the fourth; the first is to the second, as the third is to the fourth.

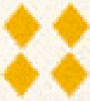
Let , the first, be the same multiple of , the second,

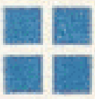
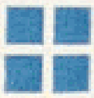
that , the third, is of , the fourth.

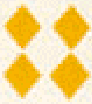
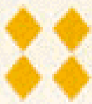
Then  :  ::  : 

take M , *m* , M , *m* ;

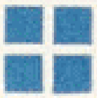
because  is the same multiple of 

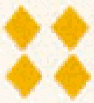
that  is of  (according to the hypothesis);

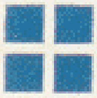
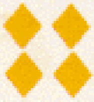
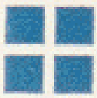
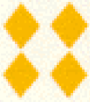
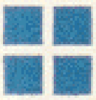
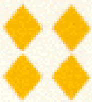
and M  is taken the same multiple of 

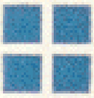
that M  is of ,

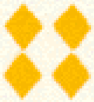
∴ (according to the third proposition),

M  is the same multiple of 

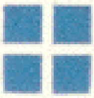
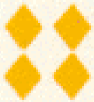
that M  is of .

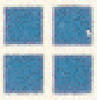
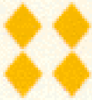
Therefore, if M  be of  a greater multiple than *m*  is, then M  is a greater multiple of  than *m*  is; that is, if M  be greater than *m* , then M  will be greater than *m* ; in the ſame manner it can be ſhewn, if M  be equal *m* , then M  will be equal *m* .

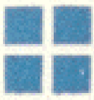
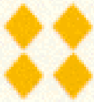
And, generally, if M  ⊏, = or ⊐ *m* 

then M  will be ⊏, = or ⊐ *m* ;

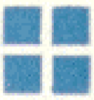
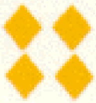
∴ by the fifth definition,

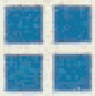
 :  ::  : .

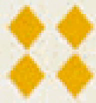
Next, let  be the ſame part of  that  is of .

In this caſe alſo  :  ::  : .

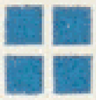
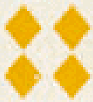
For, becauſe

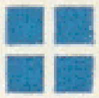
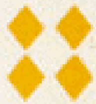
 is the ſame part of  that  is of ,

therefore  is the ſame multiple of 

that  is of .

Therefore, by the preceding caſe,

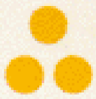
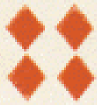
 :  ::  :  ;

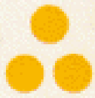
and ∴  :  ::  : ,

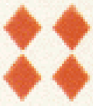
by propoſition B.

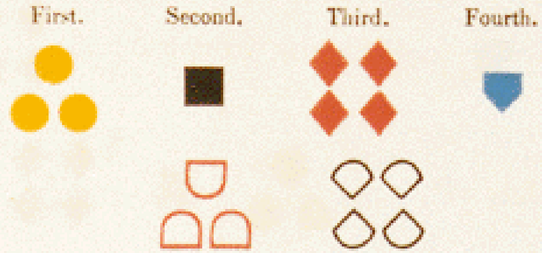
∴ If the firſt be the ſame multiple, &c.

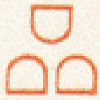
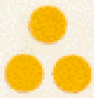
IF the firſt be to the ſecond as the third to the fourth, and if the firſt be a multiple, or a part of the ſecond; the third is the ſame multiple, or the ſame part of the fourth.

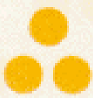
Let  :  ::  : ;

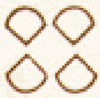
and firſt, let  be a multiple ;

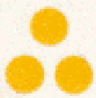
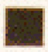
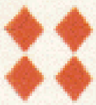
 ſhall be the ſame multiple of .



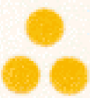
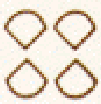
Take  = .

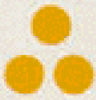
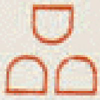
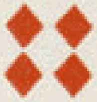
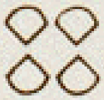
Whatever multiple  is of 

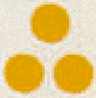
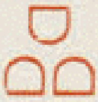
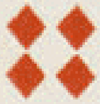
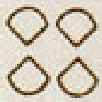
take  the ſame multiple of ,

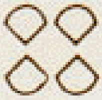
then, becauſe  :  ::  : 

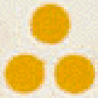
and of the ſecond and fourth, we have taken equimultiples,

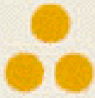
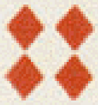
 and , therefore (B. 5. pr. 4),

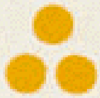
 :  ::  : , but (conſt.),

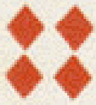
 =  ∴ (B. 5. pr. A.)  = 

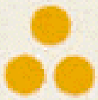
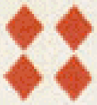
and  is the ſame multiple of 

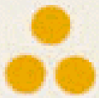
that  is of .

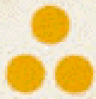
Next, let  :  ::  : ,

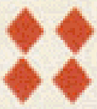
and alſo  a part of ;

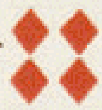
then  ſhall be the ſame part of .

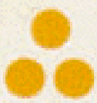
Inverſely (B. 5.),  :  ::  : ,

but  is a part of ;

that is,  is a multiple of ;

∴ by the preceding caſe,  is the ſame multiple of 

that is,  is the ſame part of 

that  is of .

∴ If the firſt be to the ſecond, &c.

EQUAL *magnitudes have the ſame ratio to the ſame magnitude, and the ſame has the ſame ratio to equal magnitudes.*

Let ● = ◆ and ■ any other magnitude;

then ● : ■ = ◆ : ■ and ■ : ● = ■ : ◆.

Becauſe ● = ◆,

∴ M ● = M ◆;

∴ if M ● ⊏, = or ⊐ *m* ■, then

M ◆ ⊏, = or ⊐ *m* ■,

and ∴ ● : ■ = ◆ : ■ (B. 5. def. 5).

From the foregoing reaſoning it is evident that,

if *m* ■ ⊏, = or ⊐ M ●, then

m ■ ⊏, = or ⊐ M ◆

∴ ■ : ● = ■ : ◆ (B. 5. def. 5).

∴ Equal magnitudes, &c.

DEFINITION VII.

When of the equimultiples of four magnitudes (taken as in the fifth definition), the multiple of the firſt is greater than that of the ſecond, but the multiple of the third is not greater than the multiple of the fourth; then the firſt is ſaid to have to the ſecond a greater ratio than the third magnitude has to the fourth: and, on the contrary, the third is ſaid to have to the fourth a leſs ratio than the firſt has to the ſecond.

If, among the equimultiples of four magnitudes, compared as in the fifth definition, we ſhould find

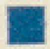
●●●●● ⊏ ■■■■, but

◆◆◆◆◆ = or ⊐ ⬟⬟⬟⬟, or if we ſhould find any particular multiple M′ of the firſt and third, and a particular multiple *m′* of the ſecond and fourth, ſuch, that M′ times the firſt is ⊏ *m′* times the ſecond, but M′ times the third is not ⊏ *m′* times the fourth, *i.e.* = or ⊐ *m′* times the fourth; then the firſt is ſaid to have to the ſecond a greater ratio than the third has to the fourth; or the third has to the fourth, under ſuch circumſtances, a leſs ratio than the firſt has to the ſecond: although ſeveral other equimultiples may tend to ſhow that the four magnitudes are proportionals.

This definition will in future be expreſſed thus:—

If M′ ⬟ ⊏ *m′* ⬠, but M′ ■ = or ⊐ *m′* ◆,

then ⬟ : ⬠ ⊏ ■ : ◆.

In the above general expreſſion, M′ and *m′* are to be conſidered particular multiples, not like the multiples M and *m* introduced in the fifth definition, which are in that definition confidered to be every pair of multiples that can be taken. It muft alfo be here obferved, that , , , and the like fymbols are to be confidered merely the reprefentatives of geometrical magnitudes.

In a partial arithmetical way, this may be fet forth as follows:

Let us take the four numbers, 8, 7, 10, and 9.

Firft.	*Second.*	*Third.*	*Fourth.*
8	7	10	9
16	14	20	18
24	21	30	27
32	28	40	36
40	35	50	45
48	42	60	54
56	49	70	63
64	56	80	72
72	63	90	81
80	70	100	90
88	77	110	99
96	84	120	108
104	91	130	117
112	98	140	126
&c.	&c.	&c	&c.

Among the above multiples we find 16 ⊏ 14 and 20 ⊏ 18; that is, twice the firft is greater than twice the fecond, and twice the third is greater than twice the fourth; and 16 ⊐ 21 and 20 ⊐ 27; that is, twice the firft is lefs than three times the fecond, and twice the third is lefs than three times the fourth; and among the fame multiples we can find 72 ⊏ 56 and 90 ⊏ 72: that is, 9 times the firft is greater than 8 times the fecond, and 9 times the third is greater than 8 times the fourth. Many other equimultiples might be selected, which would tend to ſhow that the numbers 8, 7, 10, 9, were proportionals, but they are not, for we can find a multiple of the firſt ⊏ a multiple of the ſecond, but the ſame multiple of the third that has been taken of the firſt not ⊏ the ſame multiple of the fourth which has been taken of the ſecond; for inſtance, 9 times the firſt is ⊏ 10 times the ſecond, but 9 times the third is not ⊏ 10 times the fourth, that is, 72 ⊏ 70, but 90 not ⊏ 90, or 8 times the firſt we find ⊏ 9 times the ſecond, but 8 times the third is not greater than 9 times the fourth, that is, 64 ⊏ 63, but 80 is not ⊏ 81. When any ſuch multiples as theſe can be found, the firſt (8) is ſaid to have to the ſecond (7) a greater ratio than the third (10) has to the fourth (9), and on the contrary the third (10) is ſaid to have to the fourth (9) a leſs ratio than the firſt (8) has to the ſecond (7).

Of unequal magnitudes the greater has a greater ratio to the ſame than the leſs has: and the ſame magnitude has a greater ratio to the leſs than it has to the greater.

Let ▲🟥 and 🟨 be two unequal magnitudes,
and 🔵 any other.

We ſhall firſt prove that ▲🟥 which is the greater of the two unequal magnitudes, has a greater ratio to 🔵 than 🟨, the leſs, has to 🔵;

that is, ▲🟥 : 🔵 ⊏ 🟨 : 🔵;

take M′ ▲🟥, *m′* 🔵, M′ 🟨, and *m′* 🔵;
ſuch, that M′ ▲ and M′ 🟥 ſhall be each ⊏ 🔵;
alſo take *m′* 🔵 the leaſt multiple of 🔵,
which will make *m′* 🔵 ⊏ M′ 🟨 = M′ 🟥;

∴ M′ 🟨 is not ⊏ *m′* 🔵,

but M′ ▲🟥 is ⊏ *m′* 🔵, for,
as *m′* 🔵 is the firſt multiple which firſt becomes ⊏ M′ 🟥,
than (*m′* minus 1) 🔵 or *m′* 🔵 minus 🔵 is not ⊏ M′ 🟥,
and 🔵 is not ⊏ M′ ▲,

∴ *m′* 🔵 minus 🔵 + 🔵 muſt be ⊐ M′ 🟥 + M′ ▲;

that is, *m′* 🔵 muſt be ⊐ M′ ▲🟥;

∴ M′ ▲🟥 is ⊏ *m′* 🔵; but it has been ſhown above that

M′ ■ is not ⊏ m′ ●, therefore, by the ſeventh definition,

▲■ has to ● a greater ratio than ■ : ●.

Next we ſhall prove that ● has a greater ratio to ■, the

leſs, than it has to ▲■, the greater;

or, ● : ■ ⊏ ● : ▲■.

Take m′ ●, M′ ■, m′ ●, and M′ ▲■,
the ſame as in the firſt caſe, ſuch, that
M′ ▲ and M′ ■ will be each ⊏ ●, and m′ ● the leaſt
multiple of ●, which firſt becomes greater
than M′ ■ = M′ ■.

∴ m′ ● minus ● is not ⊏ M′ ■,
and ● is not ⊏ M′ ▲; conſequently
m′ ● minus ● + ● is ⊐ M′ ■ + M′ ▲;

∴ m′ ● is ⊐ M′ ▲■, and ∴ by the ſeventh definition,

● has to ■ a greater ratio than ● has to ▲■.

∴ Of unequal magnitudes, &c.

The contrivance employed in this propoſition for finding among the multiples taken, as in the fifth definition, a multiple of the firſt greater than the multiple of the ſecond, but the ſame multiple of the third which has been taken of the firſt, not greater than the ſame multiple of the fourth which has been taken of the ſecond, may be illuſtrated numerically as follows:—

The number 9 has a greater ratio to 7 than 8 has to 7: that is, 9 : 7 ⊏ 8 : 7; or, 8 + 1 : 7 ⊏ 8 : 7.

The multiple of 1, which firſt becomes greater than 7, is 8 times, therefore we may multiply the firſt and third by 8, 9, 10, or any other greater number; in this caſe, let us multiply the firſt and third by 8, and we have 64 + 8 and 64: again, the firſt multiple of 7 which becomes greater than 64 is 10 times; then, by multiplying the ſecond and fourth by 10, we ſhall have 70 and 70; then, arranging theſe multiples, we have—

8 times the first.	10 times the second.	8 times the third.	10 times the fourth.
64 + 8	70	64	70

Conſequently 64 + 8, or 72, is greater than 70, but 64 is not greater than 70, ∴ by the ſeventh definition, 9 has a greater ratio to 7 than 8 has to 7.

The above is merely illuſtrative of the foregoing demonſtration, for this property could be ſhown of theſe or other numbers very readily in the following manner; becauſe, if an antecedent contains its conſequent a greater number of times than another antecedent contains its conſequent, or when a fraction is formed of an antecedent for the numerator, and its conſequent for the denominator be greater than another fraction which is formed of another antecedent for the numerator and its conſequent for the denominator, the ratio of the firſt antecedent to its conſequent is greater than the ratio of the laſt antecedent to its conſequent.

Thus, the number 9 has a greater ratio to 7, than 8 has to 7, for $\frac{9}{7}$ is greater than $\frac{8}{7}$.

Again, 17 : 19 is a greater ratio than 13 : 15, becauſe $\frac{17}{19} = \frac{17 \times 15}{19 \times 15} = \frac{255}{285}$, and $\frac{13}{15} = \frac{13 \times 19}{15 \times 19} = \frac{247}{285}$, hence it is evident that $\frac{255}{285}$ is greater than $\frac{247}{285}$, ∴ $\frac{17}{19}$ is greater than

$\frac{13}{15}$, and, according to what has been above ſhown, 17 has to 19 a greater ratio than 13 has to 15.

So that the general terms upon which a greater, equal, or leſs ratio exiſts are as follows:—

If $\frac{A}{B}$ be greater than $\frac{C}{D}$, A is ſaid to have to B a greater ratio than C has to D; if $\frac{A}{B}$ be equal to $\frac{C}{D}$, then A has to B the ſame ratio which C has to D; and if $\frac{A}{B}$ be leſs than $\frac{C}{D}$, A is ſaid to have to B a leſs ratio than C has to D.

The ſtudent ſhould underſtand all up to this propoſition perfectly before proceeding further, in order fully to comprehend the following propoſitions of this book. We therefore ſtrongly recommend the learner to commence again, and read up to this ſlowly, and carefully reaſon at each ſtep, as he proceeds, particularly guarding againſt the miſchievous ſyſtem of depending wholly on the memory. By following theſe inſtructions, he will find that the parts which uſually preſent conſiderable difficulties will preſent no difficulties whatever, in proſecuting the ſtudy of this important book.

MAGNITUDES *which have the ſame ratio to the ſame magnitude are equal to one another; and thoſe to which the ſame magnitude has the ſame ratio are equal to one another.*

Let ◆ : ■ :: ● : ■, then ◆ = ●.

For, if not, let ◆ ⊏ ●, then will
◆ : ■ ⊏ ● : ■ (B. 5. pr. 8),
which is abſurd according to the hypotheſis.

∴ ◆ is not ⊏ ●.

In the ſame manner it may be ſhown, that

● is not ⊏ ◆,

∴ ◆ = ●.

Again, let ■ : ◆ :: ■ : ●, then will ◆ = ●.

For (invert.) ◆ : ■ :: ● : ■,
therefore, by the firſt caſe, ◆ = ●.

∴ Magnitudes which have the ſame ratio, &c.

This may be ſhown otherwiſe, as follows :—

Let A : B = A : C, then B = C, for, as the fraction $\frac{A}{B}$ = the fraction $\frac{A}{C}$, and the numerator of one equal to the numerator of the other, therefore the denominator of theſe fractions are equal, that is B = C.

Again, if B : A = C : A, B = C. For, as $\frac{B}{A} = \frac{C}{A}$, B muſt = C.

THAT *magnitude which has a greater ratio than another has unto the ſame magnitude, is the greater of the two: and that magnitude to which the ſame has a greater ratio than it has unto another magnitude, is the leſs of the two.*

Let ⛊ : ■ ⊏ ● : ■, then ⛊ ⊏ ●.

For if not, let ⛊ = or ⊐ ●;

then, ⛊ : ■ = ● : ■ (B. 5. pr. 7) or

⛊ : ■ ⊐ ● : ■ (B. 5. pr. 8) and (invert.),

which is abſurd according to the hypotheſis.

∴ ⛊ is not = or ⊐ ●, and

∴ ⛊ muſt be ⊏ ●.

Again, let ■ : ● ⊏ ■ : ⛊,

then, ● ⊐ ⛊.

For if not, ● muſt be ⊏ or = ⛊,

then ■ : ● ⊐ ■ : ⛊ (B. 5. pr. 8) and (invert.);

or ■ : ● = ■ : ⛊ (B. 5. pr. 7), which is abſurd (hyp.);

∴ ● is not ⊏ or = ⛊,

and ∴ ● muſt be ⊐ ⛊.

∴ That magnitude which has, &c.

RATIOS *that are the ſame to the ſame ratio, are the ſame to each other.*

Let ◆ : ■ = ● : ⬟ and ● : ⬟ = ▲ : •,

then will ◆ : ■ = ▲ : •.

For if M ◆ ⊏, =, or ⊐ *m* ■,

then M ● ⊏, =, or ⊐ *m* ⬟,

and if M ● ⊏, =, or ⊐ *m* ⬟,

then M ▲ ⊏, =, or ⊐ *m* •, (B. 5. def. 5);

∴ if M ◆ ⊏, =, or ⊐ *m* ■, M ▲ ⊏, =, or ⊐ *m* •,

and ∴ (B. 5. def. 5) ◆ : ■ = ▲ : •.

∴ Ratios that are the ſame, &c.

IF any number of magnitudes be proportionals, as one of the antecedents is to its consequent, so shall all the antecedents taken together be to all the consequents.

Let ■ : ⬤ = ◡ : ⬠ = ◆ : ⬟ = • : ▾ = ▴ : ●;

then will ■ : ⬤ =

■ + ◡ + ◆ + • + ▴ : ⬤ + ⬠ + ⬟ + ▾ + ●.

For if M ■ ⊏ *m* ⬤, then M ◡ ⊏ *m* ⬠,

and M ◆ ⊏ *m* ⬟ M • ⊏ *m* ▾,

also M ▴ ⊏ *m* ●. (B. 5. def. 5.)

Therefore, if M ■ ⊏ *m* ⬤, then will

M ■ + M ◡ + M ◆ + M • + M ▴,

or M (■ + ◡ + ◆ + • + ▴) be greater

than *m* ⬤ + *m* ⬠ + *m* ⬟ + *m* ▾ + *m* ●,

or *m* (⬤ + ⬠ + ⬟ + ▾ + ●).

In the same way it may be shown, if M times one of the antecedents be equal to or less than *m* times one of the consequents, M times all the antecedents taken together, will be equal to or less than *m* times all the consequents taken together. Therefore, by the fifth definition, as one of the antecedents is to its consequent, so are all the antecedents taken together to all the consequents taken together.

∴ If any number of magnitudes, &c.

IF *the firſt has to the ſecond the ſame ratio which the third has to the fourth, but the third to the fourth a greater ratio than the fifth has to the ſixth; the firſt ſhall alſo have to the ſecond a greater ratio than the fifth to the ſixth.*

Let ⬟ : ⬠ = ■ : ◆, but ■ : ◆ ⊏ ⬡ : ●,

then ⬟ : ⬠ ⊏ ⬡ : ●.

For, becauſe ■ : ◆ ⊏ ⬡ : ●, there are ſome multiples (M′ and *m*′) of ■ and ⬡, and of ◆ and ●,

ſuch that M′ ■ ⊏ *m*′ ◆,

but M′ ⬡ not ⊏ *m*′ ●, by the ſeventh definition.

Let theſe multiples be taken, and take the ſame multiples of ⬟ and ⬠.

∴ (B. 5. def. 5.) if M′ ⬟ ⊏, =, or ⊐ *m*′ ⬠;

then will M′ ■ ⊏, =, or ⊐ *m*′ ◆,

but M′ ■ ⊏ *m*′ ◆ (conſtruction);

∴ M′ ⬟ ⊏ *m*′ ⬠,

but M′ ⬡ is not ⊏ *m*′ ● (conſtruction);

and therefore by the ſeventh definition,

⬟ : ⬠ ⊏ ⬡ : ●.

∴ If the firſt has to the ſecond, &c.

IF the firſt has the ſame ratio to the ſecond which the third has to the fourth; then, if the firſt be greater than the third, the ſecond ſhall be greater than the fourth; and if equal, equal; and if leſs, leſs.

Let ⛊ : ⛉ :: ■ : ◆, and firſt ſuppoſe
⛊ ⊏ ■, then will ⛉ ⊏ ◆.

For ⛊ : ⛉ ⊏ ■ : ⛉ (B. 5. pr. 8), and by the
hypotheſis, ⛊ : ⛉ = ■ : ◆;
∴ ■ : ◆ ⊏ ■ : ⛉ (B. 5. pr. 13),
∴ ◆ ⊐ ⛉ (B. 5. pr. 10.), or ⛉ ⊏ ◆.

Secondly, let ⛊ = ■, then will ⛉ = ◆.

For ⛊ : ⛉ = ■ : ⛉ (B. 5. pr. 7),
and ⛊ : ⛉ = ■ : ◆ (hyp.);
∴ ■ : ⛉ = ■ : ◆ (B. 5. pr. 11),
and ∴ ⛉ = ◆ (B. 5, pr. 9).

Thirdly, if ⛊ ⊐ ■, then will ⛉ ⊐ ◆;
becauſe ■ ⊏ ⛊ and ■ : ◆ = ⛊ : ⛉;
∴ ◆ ⊏ ⛉, by the firſt caſe,
that is, ⛉ ⊐ ◆.

∴ If the firſt has the ſame ratio, &c.

MAGNITUDES *have the ſame ratio to one another which their equimultiples have.*

Let ● and ■ be two magnitudes;

then, ● : ■ :: M′ ● : M′ ■.

For ● : ■ = ● : ■

= ● : ■

= ● : ■

∴ ● : ■ :: 4 ● : 4 ■. (B. 5. pr. 12).

And as the ſame reaſoning is generally applicable, we have

● : ■ :: M′ ● : M′ ■.

∴ Magnitudes have the ſame ratio, &c.

DEFINITION XIII.

THE technical term permutando, or alternando, by permutation or alternately, is uſed when there are four proportionals, and it is inferred that the firſt has the ſame ratio to the third which the ſecond has to the fourth; or that the firſt is to the third as the ſecond is to the fourth: as is ſhown in the following propoſition :—

Let ● : ◆ :: ▼ : ■,
by "permutando" or "alternando" it is
inferred ● : ▼ :: ◆ : ■.

It may be neceſſary here to remark that the magnitudes ●, ◆, ▼, ■, muſt be homogeneous, that is, of the ſame nature or ſimilitude of kind; we muſt therefore, in ſuch caſes, compare lines with lines, ſurfaces with ſurfaces, ſolids with ſolids, &c. Hence the ſtudent will readily perceive that a line and a ſurface, a ſurface and a ſolid, or other heterogenous magnitudes, can never ſtand in the relation of antecedent and conſequent.

IF *four magnitudes of the ſame kind be proportionals, they are alſo proportionals when taken alternately.*

Let ▼ : ⊔ :: ■ : ◆, then ▼ : ■ :: ⊔ : ◆.

For M ▼ : M ⊔ :: ▼ : ⊔ (B. 5. pr. 15),

and M ▼ : M ⊔ :: ■ : ◆ (hyp.) and (B. 5. pr. 11);

alſo *m* ■ : *m* ◆ :: ■ : ◆ (B. 5. pr. 15);

∴ M ▼ : M ⊔ :: *m* ■ : *m* ◆ (B. 5. pr. 14),

and ∴ if M ▼ ⊏, =, or ⊐ *m* ■,

then will M ⊔ ⊏, =, or ⊐ *m* ◆ (B. 5. pr. 14);

therefore, by the fifth definition,

▼ : ■ :: ⊔ : ◆.

∴ If four magnitudes of the ſame kind, &c.

DEFINITION XVI.

Dividendo, by divifion, when there are four proportionals, and it is inferred, that the excefs of the firft above the fecond is to the fecond, as the excefs of the third above the fourth, is to the fourth.

Let A : B :: C : D ;

by "dividendo" it is inferred

A minus B : B :: C minus D : D.

According to the above, A is fuppofed to be greater than B, and C greater than D; if this be not the cafe, but to have B greater than A, and D greater than C, B and D can be made to ftand as antecedents, and A and C as confequents, by "invertion"

B : A :: D : C ;

then, by "dividendo," we infer

B minus A : A :: D minus C : C.

IF magnitudes, taken jointly, be proportionals, they ſhall alſo be proportionals when taken ſeparately: that is, if two magnitudes together have to one of them the ſame ratio which two others have to one of theſe, the remaining one of the firſt two ſhall have to the other the ſame ratio which the remaining one of the laſt two has to the other of theſe.

Let ▼ + ⋃ : ⋃ :: ■ + ◆ : ◆,

then will ▼ : ⋃ :: ■ : ◆.

Take M ▼ ⊏ *m* ⋃ to each add M ⋃,

then we have M ▼ + M ⋃ ⊏ *m* ⋃ + M ⋃,

or M (▼ + ⋃) ⊏ (*m* + M) ⋃:

but becauſe ▼ + ⋃ : ⋃ :: ■ + ◆ : ◆ (hyp.),

and M (▼ + ⋃) ⊏ (*m* + M) ⋃;

∴ M (■ + ◆) ⊏ (*m* + M) ◆ (B. 5. def. 5);

∴ M ■ + M ◆ ⊏ *m* ◆ + M ◆;

∴ M ■ ⊏ *m* ◆, by taking M ◆ from both ſides:

that is, when M ▼ ⊏ *m* ⋃, then M ■ ⊏ *m* ◆.

In the ſame manner it may be proved, that if

M ▼ = or ⊐ *m* ⋃, then will M ■ = or ⊐ *m* ◆;

and ∴ ▼ : ⋃ :: ■ : ◆ (B. 5. def. 5).

∴ If magnitudes taken jointly, &c.

DEFINITION XV.

The term componendo, by compofition, is ufed when there are four proportionals; and it is inferred that the firft together with the fecond is to the fecond as the third together with the fourth is to the fourth.

Let A : B :: C : D;

then, by the term "componendo," it is inferred that

A + B : B :: C + D : D.

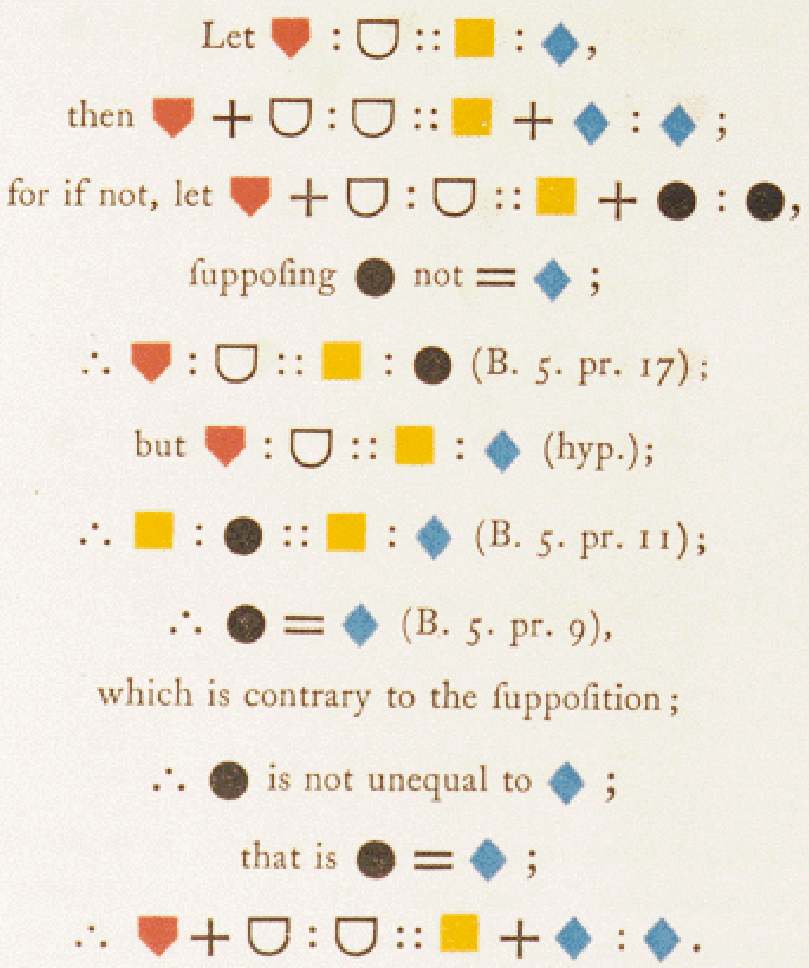
By "invertion" B and D may become the firft and third, A and C the fecond and fourth, as

B : A :: D : C,

then, by "componendo," we infer that

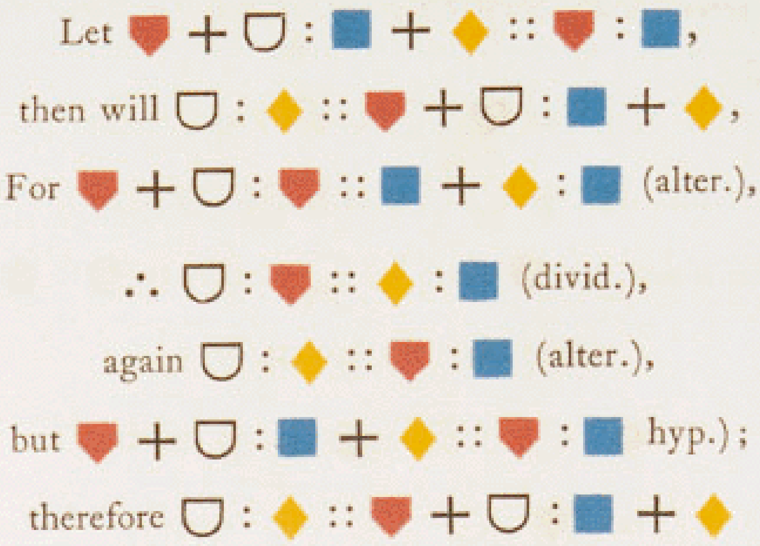
B + A : A :: D + C : C.

IF magnitudes, taken ſeparately, be proportionals, they ſhall alſo be proportionals when taken jointly: that is, if the firſt be to the ſecond as the third is to the fourth, the firſt and ſecond together ſhall be to the ſecond as the third and fourth together is to the fourth.



∴ If magnitudes, taken ſeparately, &c.

IF a whole magnitude be to a whole, as a magnitude taken from the firſt, is to a magnitude taken from the other; the remainder ſhall be to the remainder, as the whole to the whole.



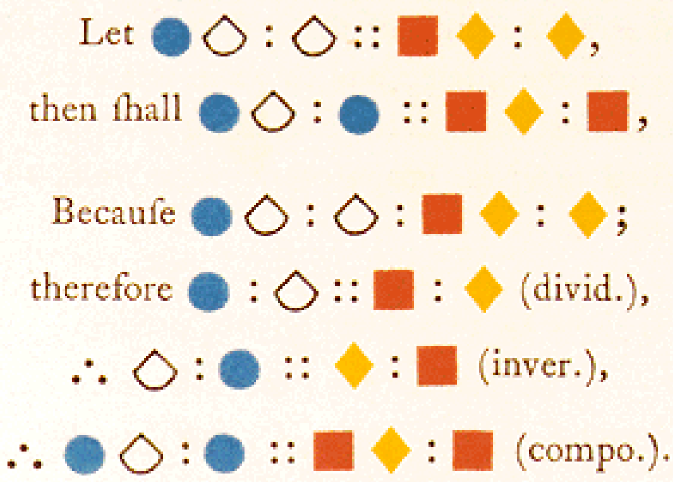
(B. 5. pr. 11).

∴ If a whole magnitude be to a whole, &c.

DEFINITION XVII.

THE term "convertendo," by converſion, is made uſe of by geometricians, when there are four proportionals, and it is inferred, that the firſt is to its exceſs above the ſecond, as the third is to its exceſs above the fourth. See the following propoſition:—

IF four magnitudes be proportionals, they are alſo proportionals by converſion: that is, the firſt is to its exceſs above the ſecond, as the third to its exceſs above the fourth.



∴ If four magnitudes, &c.

DEFINITION XVIII.

"Ex æquali" (ſc. diſtantiâ), or ex æquo, from equality of diſtance: when there is any number of magnitudes more than two, and as many others, ſuch that they are proportionals when taken two and two of each rank, and it is inferred that the firſt is to the laſt of the firſt rank of magnitudes, as the firſt is to the laſt of the others: "of this there are the two following kinds, which ariſe from the different order in which the magnitudes are taken, two and two."

DEFINITION XIX.

" Ex æquali," from equality. This term is uſed ſimply by itſelf, when the firſt magnitude is to the ſecond of the firſt rank, as the firſt to the ſecond of the other rank; and as the ſecond is to the third of the firſt rank, ſo is the ſecond to the third of the other; and ſo on in order: and the inference is as mentioned in the preceding definition; whence this is called ordinate proportion. It is demonſtrated in Book 5. pr. 22.

Thus, if there be two ranks of magnitudes,

A, B, C, D, E, F, the firſt rank,

and L, M, N, O, P, Q, the ſecond,

ſuch that A : B :: L : M, B : C :: M : N,

C : D :: N : O, D : E :: O : P, E : F :: P : Q;

we infer by the term "ex æquali" that

A : F :: L : Q.

DEFINITION XX.

"Ex æquali in proportione perturbatâ ſeu inordinatâ," from equality in perturbate, or diſorderly proportion. This term is uſed when the firſt magnitude is to the ſecond of the firſt rank as the laſt but one is to the laſt of the ſecond rank; and as the ſecond is to the third of the firſt rank, ſo is the laſt but two to the laſt but one of the ſecond rank; and as the third is to the fourth of the firſt rank, ſo is the third from the laſt to the laſt but two of the ſecond rank; and ſo on in a croſs order: and the inference is in the 18th definition. It is demonſtrated in B. 5. pr. 23.

Thus, if there be two ranks of magnitudes,

A, B, C, D, E, F, the firſt rank,

and L, M, N, O, P, Q, the ſecond,

ſuch that A : B :: P : Q, B : C :: O : P,

C : D :: N : O, D : E :: M : N, E : F :: L : M;

the term "ex æquali in proportione perturbatâ ſeu inordinatâ" infers that

A : F :: L : Q.

IF there be three magnitudes, and other three, which, taken two and two, have the same ratio; then, if the first be greater than the third, the fourth shall be greater than the sixth; and if equal, equal; and if less, less.

Let ⬟, ◡, ■, be the first three magnitudes,

and ◆, ⬠, ●, be the other three,

such that ⬟ : ◡ :: ◆ : ⬠, and ◡ : ■ :: ⬠ : ●.

Then, if ⬟ ⊏, =, or ⊐ ■, then will ◆ ⊏, =, or ⊐ ●.

From the hypothesis, by alternando, we have

⬟ : ◆ :: ◡ : ⬠,

and ◡ : ⬠ :: ■ : ●;

∴ ⬟ : ◆ :: ■ : ● (B. 5. pr. 11);

∴ if ⬟ ⊏, =, or ⊐ ■, then will ◆ ⊏, =, or ⊐ ● (B. 5. pr. 14).

∴ If there be three magnitudes, &c.

IF there be three magnitudes, and other three which have the ſame ratio, taken two and two, but in a croſs order; then if the firſt magnitude be greater than the third, the fourth ſhall be greater than the ſixth; and if equal, equal; and if leſs, leſs.

Let ⬟, ⌂, ■, be the firſt three magnitudes,

and ◆, ⬠, ●, the other three,

ſuch that ⬟ : ⌂ :: ⬠ : ●, and ⌂ : ■ :: ◆ : ⬠.

Then, if ⬟ ⊏, =, or ⊐ ■, then

will ◆ ⊏, =, ⊐ ●.

Firſt, let ⬟ be ⊏ ■ :

then, becauſe ⌂ is any other magnitude,

⬟ : ⌂ ⊏ ■ : ⌂ (B. 5. pr. 8);

but ⬠ : ● :: ⬟ : ⌂ (hyp.);

∴ ⬠ : ● ⊏ ■ : ⌂ (B. 5. pr. 13);

and becauſe ⌂ : ■ :: ◆ : ⬠ (hyp.);

∴ ■ : ⌂ :: ⬠ : ◆ (inv.),

and it was ſhown that ⬠ : ● ⊏ ■ : ⌂,

∴ ⬠ : ● ⊏ ⬠ : ◆ (B. 5. pr. 13);

∴ ● ⊐ ◆,

that is ◆ ⊏ ●.

Secondly, let ▼ = ■; then ſhall ◆ = ●.

For becauſe ▼ = ■,

▼ : ▲ = ■ : ▲ (B. 5. pr. 7);

but ▼ : ▲ = ◇ : ● (hyp.),

and ■ : ▲ = ◇ : ◆ (hyp. and inv.),

∴ ◇ : ● = ◇ : ◆ (B. 5. pr. 11),

∴ ◆ = ● (B. 5. pr. 9).

Next, let ▼ be ⊐ ■, then ◆ ſhall be ⊐ ●;

for ■ ⊏ ▼,

and it has been ſhown that ■ : ▲ = ◇ : ◆,

and ▲ : ▼ = ● : ◇;

∴ by the firſt caſe ● is ⊏ ◆,

that is, ◆ ⊐ ●.

∴ If there be three, &c.

If there be any number of magnitudes, and as many others, which, taken two and two in order, have the ſame ratio; the firſt ſhall have to the laſt of the firſt magnitudes the ſame ratio which the firſt of the others has to the laſt of the ſame.

N.B.—*This is uſually cited by the words "ex æquali," or "ex æquo."*

Firſt, let there be magnitudes ⬟, ◆, ■,
and as many others ♦, ◇, ●,
ſuch that
⬟ : ◆ :: ♦ : ◇,
and ◆ : ■ :: ◇ : ●;
then ſhall ⬟ : ■ :: ♦ : ●.

Let theſe magnitudes, as well as any equimultiples whatever of the antecedents and conſequents of the ratios, ſtand as follows :—

⬟, ◆, ■, ♦, ◇, ●,
and
M ⬟, *m* ◆, N ■, M ♦, *m* ◇, N ●,
becauſe ⬟ : ◆ :: ♦ : ◇;
∴ M ⬟ : *m* ◆ :: M ♦ : *m* ◇ (B. 5. p. 4).

For the ſame reaſon
m ◆ : N ■ :: *m* ◇ : N ●;
and becauſe there are three magnitudes,

M ⬟, *m* ◆, N ■,

and other three, M ◆, *m* ◇, N ●,

which, taken two and two, have the ſame ratio;

∴ if M ⬟ ⊏, =, or ⊐ N ■

then will M ◆ ⊏, =, or ⊐ N ●, by (B. 5. pr. 20);

and ∴ ⬟ : ■ :: ◆ : ● (def. 5).

Next, let there be four magnitudes, ⬟, ◆, ■, ◆,

and other four, ◇, ●, ▬, ▲,

which, taken two and two, have the ſame ratio,

that is to ſay, ⬟ : ◆ :: ◇ : ●,

◆ : ■ :: ● : ▬,

and ■ : ◆ :: ▬ : ▲,

then ſhall ⬟ : ◆ :: ◇ : ▲;

for, becauſe ⬟, ◆, ■, are three magnitudes,

and ◇, ●, ▬, other three,

which, taken two and two, have the ſame ratio;

therefore, by the foregoing caſe, ⬟ : ■ :: ◇ : ▬,

but ■ : ◆ :: ▬ : ▲;

therefore again, by the firſt caſe, ⬟ : ◆ :: ◇ : ▲;

and ſo on, whatever the number of magnitudes be.

∴ If there be any number, &c.

IF there be any number of magnitudes, and as many others, which, taken two and two in a croſs order, have the ſame ratio; the firſt ſhall have to the laſt of the firſt magnitudes the ſame ratio which the firſt of the others has to the laſt of the ſame.

N.B.—*This is uſually cited by the words "ex æquali in proportione perturbatâ;" or "ex æquo perturbato."*

Firſt, let there be three magnitudes, 

and other three, 

which, taken two and two in a croſs order,

have the ſame ratio;

that is, 

and 

then ſhall 

Let theſe magnitudes and their reſpective equimultiples be arranged as follows:—

then  (B. 5. pr. 15);

and for the ſame reaſon

but  (hyp.),

∴ M ⬟ : M ⬠ :: ◇ : ● (B. 5. pr. 11);

and becauſe ⬠ : ■ :: ◆ : ◇ (hyp.),

∴ M ⬠ : *m* ■ :: M ◆ : *m* ◇ (B. 5. pr. 4);

then, becauſe there are three magnitudes,

M ⬟, M ⬠, *m* ■,

and other three, M ◆, *m* ◇, *m* ●,

which, taken two and two in a croſs order, have

the ſame ratio;

therefore, if M ⬟ ⊏, =, or ⊐ *m* ■,

then will M ◆ ⊏, =, or ⊐ *m* ● (B. 5. pr. 21),

and ∴ ⬟ : ■ :: ◆ : ● (B. 5. def. 5).

Next, let there be four magnitudes,

⬟, ⬠, ■, ◆,

and other four, ◇, ●, ▬, ▲,

which, when taken two and two in a croſs order, have

the ſame ratio; namely,

⬟ : ⬠ :: ▬ : ▲,

⬠ : ■ :: ● : ▬,

and ■ : ◆ :: ◇ : ●,

then ſhall ⬟ : ◆ :: ◇ : ▲.

For, becauſe ⬟, ⬠, ■ are three magnitudes, and , , , other three,

which, taken two and two in a crofs order, have the fame ratio,

therefore, by the firft cafe, : :: : ,

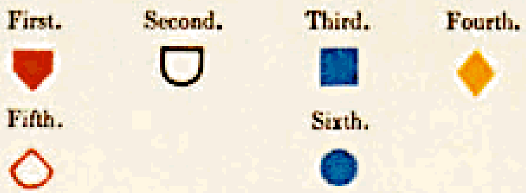
but : :: : ,

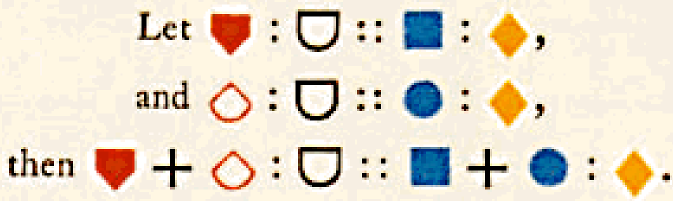
therefore again, by the firft cafe, : :: : ;

and fo on, whatever be the number of fuch magnitudes.

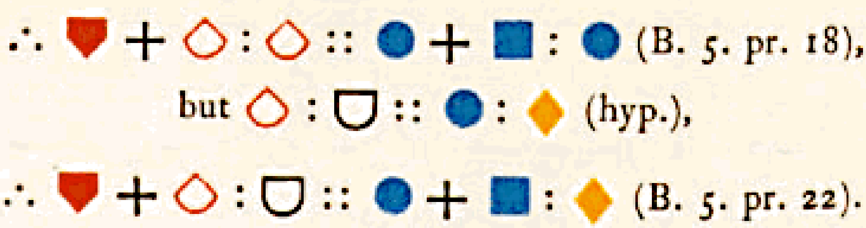
∴ If there be any number, &c.

IF the firſt has to the ſecond the ſame ratio which the third has to the fourth, and the fifth to the ſecond the ſame which the ſixth has to the fourth, the firſt and fifth together ſhall have to the ſecond the ſame ratio which the third and ſixth together have to the fourth.





and, becauſe theſe magnitudes are proportionals, they are proportionals when taken jointly,



∴ If the firſt, &c.

IF four magnitudes of the fame kind are proportionals, the greateft and leaft of them together are greater than the other two together.

Let four magnitudes, [red] + [open], [blue] + [yellow], [open], and [yellow], of the fame kind, be proportionals, that is to fay,

[red] + [open] : [blue] + [yellow] :: [open] : [yellow],

and let [red] + [open] be the greateft of the four, and confequently by pr. A and 14 of Book 5, [yellow] is the leaft;

then will [red] + [open] + [yellow] be ⊏ [blue] + [yellow] + [open];

becaufe [red] + [open] : [blue] + [yellow] :: [open] : [yellow],

∴ [red] : [blue] :: [red] + [open] : [blue] + [yellow] (B. 5. pr. 19),

but [red] + [open] ⊏ [blue] + [yellow] (hyp.),

∴ [red] ⊏ [blue] (B. 5. pr. A);

to each of thefe add [open] + [yellow],

∴ [red] + [open] + [yellow] ⊏ [blue] + [open] + [yellow].

∴ If four magnitudes, &c.

DEFINITION X.

When three magnitudes are proportionals, the firſt is ſaid to have to the third the duplicate ratio of that which it has to the ſecond.

For example, if A, B, C, be continued proportionals, that is, A : B :: B : C, A is ſaid to have to C the duplicate ratio of A : B;

$$\text{or } \frac{A}{C} = \text{the ſquare of } \frac{A}{B}.$$

This property will be more readily ſeen of the quantities ar^2, ar, a, for $ar^2 : ar :: ar : a$;

$$\text{and } \frac{ar^2}{a} = r^2 = \text{the ſquare of } \frac{ar^2}{ar} = r,$$

or of a, ar, ar^2;

$$\text{for } \frac{a}{ar^2} = \frac{1}{r^2} = \text{the ſquare of } \frac{a}{ar} = \frac{1}{r}.$$

DEFINITION XI.

When four magnitudes are continual proportionals, the firſt is ſaid to have to the fourth the triplicate ratio of that which it has to the ſecond; and ſo on, quadruplicate, &c. increaſing the denomination ſtill by unity, in any number of proportionals.

For example, let A, B, C, D, be four continued proportionals, that is, A : B :: B : C :: C : D; A is ſaid to have to D, the triplicate ratio of A to B;

$$\text{or } \frac{A}{D} = \text{the cube of } \frac{A}{B}.$$

This definition will be better underſtood, and applied to a greater number of magnitudes than four that are continued proportionals, as follows:—

Let ar^3, ar^2, ar, a, be four magnitudes in continued proportion, that is, $ar^3 : ar^2 :: ar^2 : ar :: ar : a$,

then $\frac{ar^3}{a} = r^3 =$ the cube of $\frac{ar^3}{ar^2} = r$.

Or, let ar^5, ar^4, ar^3, ar^2, ar, a, be ſix magnitudes in proportion, that is

$ar^5 : ar^4 :: ar^4 \cdot ar^3 :: ar^3 : ar^2 :: ar^2 : ar :: ar : a$,

then the ratio $\frac{ar^5}{a} = r^5 =$ the fifth power of $\frac{ar^5}{ar^4} = r$.

Or, let a, ar, ar^2, ar^3, ar^4, be five magnitudes in continued proportion; then $\frac{a}{ar^4} = \frac{1}{r^4} =$ the fourth power of $\frac{a}{ar} = \frac{1}{r}$.

DEFINITION A.

To know a compound ratio:—

When there are any number of magnitudes of the ſame kind, the firſt is ſaid to have to the laſt of them the ratio compounded of the ratio which the firſt has to the ſecond, and of the ratio which the ſecond has to the third, and of the ratio which the third has to the fourth; and ſo on, unto the laſt magnitude.

A B C D
E F G H K L
M N

For example, if A, B, C, D, be four magnitudes of the ſame kind, the firſt A is ſaid to have to the laſt D the ratio compounded of the ratio of A to B, and of the ratio of B to C, and of the ratio of C to D; or, the ratio of

A to D is ſaid to be compounded of the ratios of A to B, B to C, and C to D.

And if A has to B the ſame ratio which E has to F, and B to C the ſame ratio that G has to H, and C to D the ſame that K has to L; then by this definition, A is said to have to D the ratio compounded of ratios which are the ſame with the ratios of E to F, G to H, and K to L. And the ſame thing is to be underſtood when it is more briefly expreſſed by ſaying, A has to D the ratio compounded of the ratios of E to F, G to H, and K to L.

In like manner, the ſame things being ſuppoſed; if M has to N the ſame ratio which A has to D, then for ſhortneſs ſake, M is ſaid to have to N the ratio compounded of the ratios of E to F, G to H, and K to L.

This definition may be better underſtood from an arithmetical or algebraical illuſtration; for, in fact, a ratio compounded of ſeveral other ratios, is nothing more than a ratio which has for its antecedent the continued product of all the antecedents of the ratios compounded, and for its conſequent the continued product of all the conſequents of the ratios compounded.

Thus, the ratio compounded of the ratios of

$$2 : 3,\ 4 : 7,\ 6 : 11,\ 2 : 5,$$

is the ratio of $2 \times 4 \times 6 \times 2 : 3 \times 7 \times 11 \times 5$,

or the ratio of $96 : 1155$, or $32 : 385$.

And of the magnitudes A, B, C, D, E, F, of the ſame kind, A : F is the ratio compounded of the ratios of

$$A : B,\ B : C,\ C : D,\ D : E,\ E : F;$$

for $A \times B \times C \times D \times E : B \times C \times D \times E \times F$,

or $\frac{A \times B \times C \times D \times E}{B \times C \times D \times E \times F} = \frac{A}{F}$, or the ratio of $A : F$.

RATIOS *which are compounded of the ſame ratios are the ſame to one another.*

Let A : B :: F : G,
B : C :: G : H,
C : D :: H : K,
and D : E :: K : L.

A B C D E
F G H K L

Then the ratio which is compounded of the ratios of A : B, B : C, C : D, D : E, or the ratio of A : E, is the ſame as the ratio compounded of the ratios of F : G, G : H, H : K, K : L, or the ratio of F : L.

For $\frac{A}{B} = \frac{F}{G}$,

$\frac{B}{C} = \frac{G}{H}$,

$\frac{C}{D} = \frac{H}{K}$,

and $\frac{D}{E} = \frac{K}{L}$;

$$\therefore \frac{A \times B \times C \times D}{B \times C \times D \times E} = \frac{F \times G \times H \times K}{G \times H \times K \times L},$$

and $\therefore \frac{A}{E} = \frac{F}{L}$,

or the ratio of A : E is the ſame as the ratio of F : L.

The ſame may be demonſtrated of any number of ratios ſo circumſtanced.

Next, let A : B :: K : L,
B : C :: H : K,
C : D :: G : H,
D : E :: F : G.

Then the ratio which is compounded of the ratios of A : B, B : C, C : D, D : E, or the ratio of A : E, is the ſame as the ratio compounded of the ratios of K : L, H : K, G : H, F : G, or the ratio of F : L.

For $\frac{A}{B} = \frac{K}{L}$,

$\frac{B}{C} = \frac{H}{K}$,

$\frac{C}{D} = \frac{G}{H}$,

and $\frac{D}{E} = \frac{F}{G}$;

$$\therefore \frac{A \times B \times C \times D}{B \times C \times D \times E} = \frac{K \times H \times G \times F}{L \times K \times H \times G},$$

and $\therefore \frac{A}{E} = \frac{F}{L}$,

or the ratio of A : E is the ſame as the ratio of F : L.

∴ Ratios which are compounded, &c.

If several ratios be the same to several ratios, each to each, the ratio which is compounded of ratios which are the same to the first ratios, each to each, shall be the same to the ratio compounded of ratios which are the same to the other ratios, each to each.

A B C D E F G H	P Q R S T
a b c d e f g h	V W X Y Z

If $A : B :: a : b$ | and $A : B :: P : Q$ | $a : b :: V : W$
$C : D :: c : d$ | $C : D :: Q : R$ | $c : d :: W : X$
$E : F :: e : f$ | $E : F :: R : S$ | $e : f :: X : Y$
and $G : H :: g : h$ | $G : H :: S : T$ | $g : h :: Y : Z$

then $P : T = V : Z$.

For $\frac{P}{Q} = \frac{A}{B} = \frac{a}{b} = \frac{V}{W}$,

$\frac{Q}{R} = \frac{C}{D} = \frac{c}{d} = \frac{W}{X}$,

$\frac{R}{S} = \frac{E}{F} = \frac{e}{f} = \frac{X}{Y}$,

$\frac{S}{T} = \frac{G}{H} = \frac{g}{h} = \frac{Y}{Z}$;

and ∴ $\frac{P \times Q \times R \times S}{Q \times R \times S \times T} = \frac{V \times W \times X \times Y}{W \times X \times Y \times Z}$,

and ∴ $\frac{P}{T} = \frac{V}{Z}$,

or $P : T = V : Z$.

∴ If several ratios, &c.

If a ratio which is compounded of ſeveral ratios be the ſame to a ratio which is compounded of ſeveral other ratios; and if one of the firſt ratios, or the ratio which is compounded of ſeveral of them, be the ſame to one of the laſt ratios, or to the ratio which is compounded of ſeveral of them; then the remaining ratio of the firſt, or, if there be more than one, the ratio compounded of the remaining ratios, ſhall be the ſame to the remaining ratio of the laſt, or, if there be more than one, to the ratio compounded of theſe remaining ratios.

A B C D E F G H
P Q R S T X

Let A : B, B : C, C : D, D : E, E : F, F : G, G : H, be the firſt ratios, and P : Q, Q : R, R : S, S : T, T : X, the other ratios; alſo, let A : H, which is compounded of the firſt ratios, be the ſame as the ratio of P : X, which is the ratio compounded of the other ratios; and, let the ratio of A : E, which is compounded of the ratios of A : B, B : C, C : D, D : E, be the ſame as the ratio of P : R, which is compounded of the ratios P : Q, Q : R.

Then the ratio which is compounded of the remaining firſt ratios, that is, the ratio compounded of the ratios E : F, F : G, G : H, that is, the ratio of E : H, ſhall be the ſame as the ratio of R : X, which is compounded of the ratios of R : S, S : T, T : X, the remaining other ratios.

Becaufe $\frac{A \times B \times C \times D \times E \times F \times G}{B \times C \times D \times E \times F \times G \times H} = \frac{P \times Q \times R \times S \times T}{Q \times R \times S \times T \times X}$,

or $\frac{A \times B \times C \times D}{B \times C \times D \times E} \times \frac{E \times F \times G}{F \times G \times H} = \frac{P \times Q}{Q \times R} \times \frac{R \times S \times T}{S \times T \times X}$,

and $\frac{A \times B \times C \times D}{B \times C \times D \times E} = \frac{P \times Q}{Q \times R}$,

$\therefore \frac{E \times F \times G}{F \times G \times H} = \frac{R \times S \times T}{S \times T \times X}$,

$\therefore \frac{E}{H} = \frac{R}{X}$,

$\therefore E : H = R : X$.

∴ If a ratio which, &c.

IF there be any number of ratios, and any number of other ratios, ſuch that the ratio which is compounded of ratios, which are the ſame to the firſt ratios, each to each, is the ſame to the ratio which is compounded of ratios, which are the ſame, each to each, to the laſt ratios—and if one of the firſt ratios, or the ratio which is compounded of ratios, which are the ſame to ſeveral of the firſt ratios, each to each, be the ſame to one of the laſt ratios, or to the ratio which is compounded of ratios, which are the ſame, each to each, to ſeveral of the laſt ratios—then the remaining ratio of the firſt; or, if there be more than one, the ratio which is compounded of ratios, which are the ſame, each to each, to the remaining ratios of the firſt, ſhall be the ſame to the remaining ratio of the laſt; or, if there be more than one, to the ratio which is compounded of ratios, which are the ſame, each to each, to theſe remaining ratios.

h k m n s
AB, CD, EF, GH, KL, MN, *a b c d e f g*
OP, QR, ST, VW, XY, *h k l m n p*
a b c d e f g

Let A : B, C : D, E : F, G : H, K : L, M : N, be the firſt ratios, and O : P, Q : R, S : T, V : W, X : Y, the other ratios;

and let A : B = *a* : *b*,
C : D = *b* : *c*,
E : F = *c* : *d*,
G : H = *d* : *e*,
K : L = *e* : *f*,
M : N = *f* : *g*.

Then, by the definition of a compound ratio, the ratio of $a:g$ is compounded of the ratios of $a:b$, $b:c$, $c:d$, $d:e$, $e:f$, $f:g$, which are the ſame as the ratio of A : B, C : D, E : F, G : H, K : L, M : N, each to each.

$$\begin{aligned} \text{Alſo, } O : P &= h:k, \\ Q : R &= k:l, \\ S : T &= l:m, \\ V : W &= m:n, \\ X : Y &= n:p. \end{aligned}$$

Then will the ratio of $h:p$ be the ratio compounded of the ratios of $h:k$, $k:l$, $l:m$, $m:n$, $n:p$, which are the ſame as the ratios of O : P, Q : R, S : T, V : W, X : Y, each to each.

$$\therefore \text{ by the hypotheſis } a:g = h:p.$$

Alſo, let the ratio which is compounded of the ratios of A : B, C : D, two of the firſt ratios (or the ratios of $a:c$, for A : B $= a:b$, and C : D $= b:c$), be the ſame as the ratio of a : d, which is compounded of the ratios of a : b, b : c, c : d, which are the ſame as the ratios of O : P, Q : R, S : T, three of the other ratios.

And let the ratios of h : s, which is compounded of the ratios of h : k, k : m, m : n, n : s, which are the ſame as the remaining firſt ratios, namely, E : F, G : H, K : L, M : N ; alſo, let the ratio of e : g, be that which is compounded of the ratios e : f, f : g, which are the ſame, each to each, to the remaining other ratios, namely, V : W, X : Y. Then the ratio of h : s ſhall be the ſame as the ratio of e : g; or h : s = e : g.

$$\text{For } \frac{A \times C \times E \times G \times K \times M}{B \times D \times F \times H \times L \times N} = \frac{a \times b \times c \times d \times e \times f}{b \times c \times d \times e \times f \times g},$$

and $\frac{O \times Q \times S \times V \times X}{P \times R \times T \times W \times Y} = \frac{h \times k \times l \times m \times n}{k \times l \times m \times n \times p}$,

by the compofition of the ratios;

$$\therefore \frac{a \times b \times c \times d \times e \times f}{b \times c \times d \times e \times f \times g} = \frac{h \times k \times l \times m \times n}{k \times l \times m \times n \times p} \text{ (hyp.)},$$

$$\text{or } \frac{a \times b}{b \times c} \times \frac{c \times d \times e \times f}{d \times e \times f \times g} = \frac{h \times k \times l}{k \times l \times m} \times \frac{m \times n}{n \times p},$$

$$\text{but } \frac{a \times b}{b \times c} = \frac{A \times C}{B \times D} = \frac{O \times Q \times S}{P \times R \times T} = \frac{a \times b \times c}{b \times c \times d} = \frac{h \times k \times l}{k \times l \times m};$$

$$\therefore \frac{c \times d \times e \times f}{d \times e \times f \times g} = \frac{m \times n}{n \times p}.$$

$$\text{And } \frac{c \times d \times e \times f}{d \times e \times f \times g} = \frac{h \times k \times m \times n}{k \times m \times n \times s} \text{ (hyp.)},$$

$$\text{and } \frac{m \times n}{n \times p} = \frac{e \times f}{f \times g} \text{ (hyp.)},$$

$$\therefore \frac{h \times k \times m \times n}{k \times m \times n \times s} = \frac{e\,f}{f\,g},$$

$$\therefore \frac{h}{s} = \frac{e}{g},$$

$$\therefore h : s = e : g.$$

∴ If there be any number, &c.

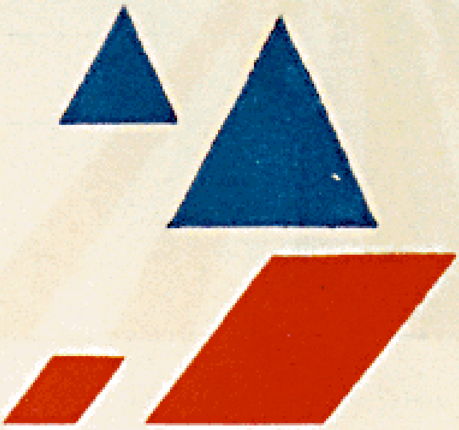
, Algebraical and Arithmetical expositions of the Fifth Book of Euclid are given in Byrne's Doctrine of Proportion; published by WILLIAMS and Co. London. 1841.

BOOK VI.

DEFINITIONS.

I.

RECTILINEAR figures are ſaid to be ſimilar, when they have their ſeveral angles equal, each to each, and the ſides about the equal angles proportional.



II.

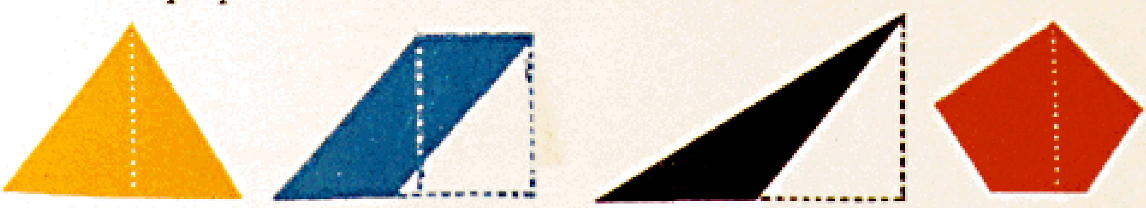
Two ſides of one figure are ſaid to be reciprocally proportional to two ſides of another figure when one of the ſides of the firſt is to the ſecond, as the remaining ſide of the ſecond is to the remaining ſide of the firſt.

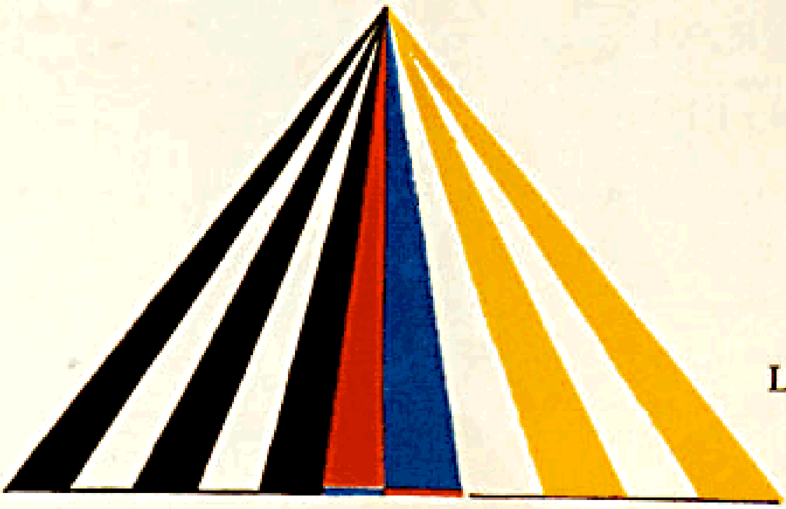
III.

A STRAIGHT line is ſaid to be cut in extreme and mean ratio, when the whole is to the greater ſegment, as the greater ſegment is to the leſs.

IV.

THE altitude of any figure is the ſtraight line drawn from its vertex perpendicular to its baſe, or the baſe produced.





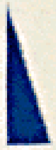
TRIANGLES *and parallelograms having the same altitude are to one another as their bases.*

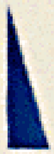
Let the triangles  and  have a common vertex, and their bases and in the same straight line.

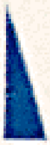
Produce both ways, take successively on produced lines equal to it; and on produced lines successively equal to it; and draw lines from the common vertex to their extremities.

The triangles  thus formed are all equal to one another, since their bases are equal. (B. 1. pr. 38.)

∴  and its base are respectively equimultiples of  and the base .

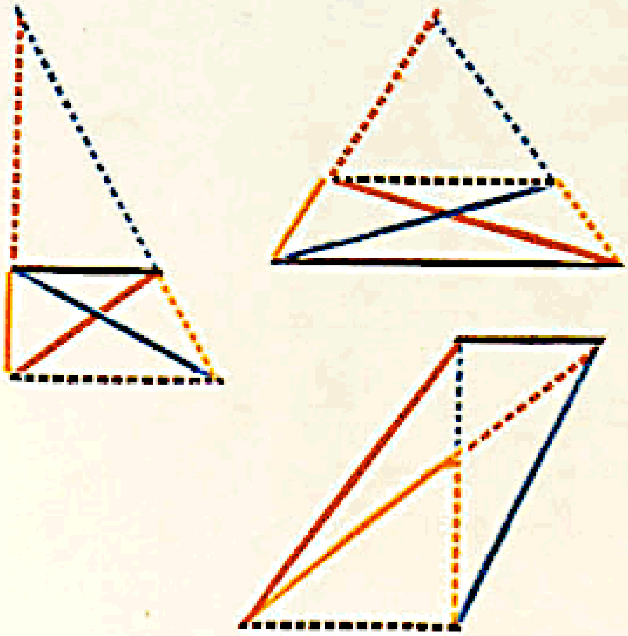
In like manner  and its baſe are reſpectively equimultiples of  and the baſe .

∴ If *m* or 6 times  ⊏ = or ⊐ *n* or 5 times  then *m* or 6 times ⊏ = or ⊐ *n* or 5 times , *m* and *n* ſtand for every multiple taken as in the fifth definition of the Fifth Book. Although we have only ſhown that this property exiſts when *m* equal 6, and *n* equal 5, yet it is evident that the property holds good for every multiple value that may be given to *m*, and to *n*.

∴  :  :: : (B. 5. def. 5.)

Parallelograms having the ſame altitude are the doubles of the triangles, on their baſes, and are proportional to them (Part 1), and hence their doubles, the parallelograms, are as their baſes. (B. 5. pr. 15.)

Q. E. D.



If a straight line  *be drawn parallel to any side* *of a triangle, it shall cut the other sides, or those sides produced, into proportional segments.*

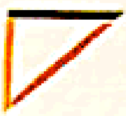
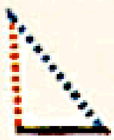
And if any straight line *divide the sides of a triangle, or those sides produced, into proportional segments, it is parallel to the remaining side* .

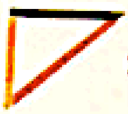
PART I.

Let || , then shall
 : :: : .

Draw and ,

and  =  (B. 1. pr. 37);

∴  :  ::  :  (B. 5. pr. 7); but

 :  :: : (B. 6. pr. 1),

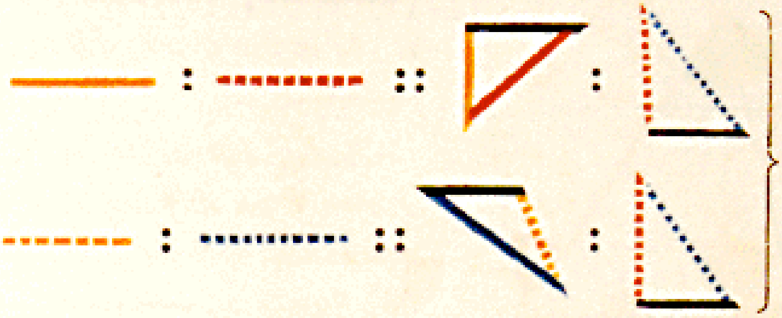
∴ : :: : .
(B. 5. pr. 11).

PART II.

Let 

then 

Let the same construction remain,

because / and  (B. 6. pr. 1);

but  (hyp.),

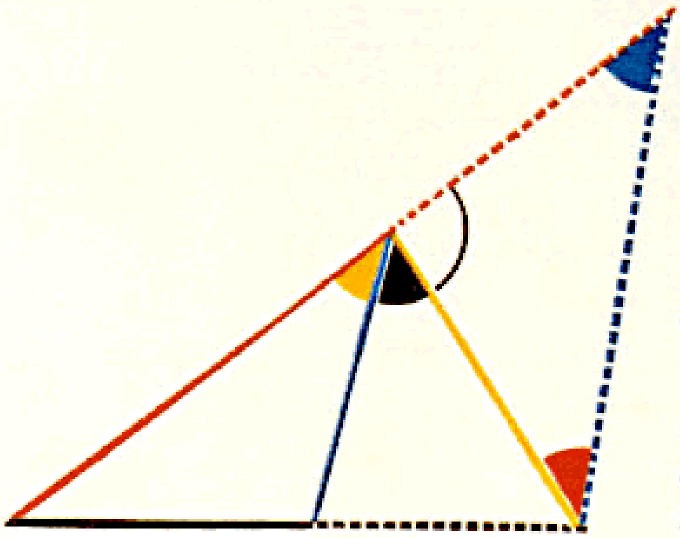
∴  (B. 5. pr. 11.)

∴  (B. 5. pr. 9);

but they are on the same base , and at the same side of it, and

∴  (B. 1. pr. 39).

Q. E. D.

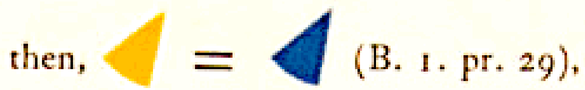


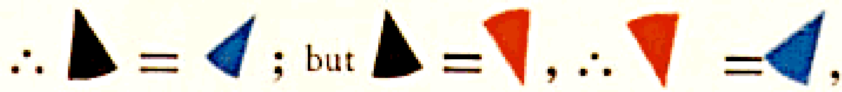
A RIGHT *line* (——) *bisecting the angle of a triangle, divides the opposite side into segments* (——, - - - -) *proportional to the conterminous sides* (——, ——).

And if a straight line (——) *drawn from any angle of a triangle divide the opposite side* (——- - -) *into segments* (——, - - - -) *proportional to the conterminous sides* (——, ——), *it bisects the angle.*

PART I.

Draw - - - - || ——, to meet - - - - ;





∴ - - - - = —— (B. 1. pr. 6);

and because —— || - - - -,

- - - - : —— :: - - - - : ——

(B. 6. pr. 2);

but - - - - = ——;

∴ —— : —— :: - - - - : ——

(B. 5. pr. 7).

PART II.

Let the ſame conſtruction remain,

and : :: :

(B. 6. pr. 2);

but : :: : (hyp.)

∴ : :: :

(B. 5. pr. 11).

and ∴ = (B. 5. pr. 9),

and ∴ = (B. 1. pr. 5); but ſince

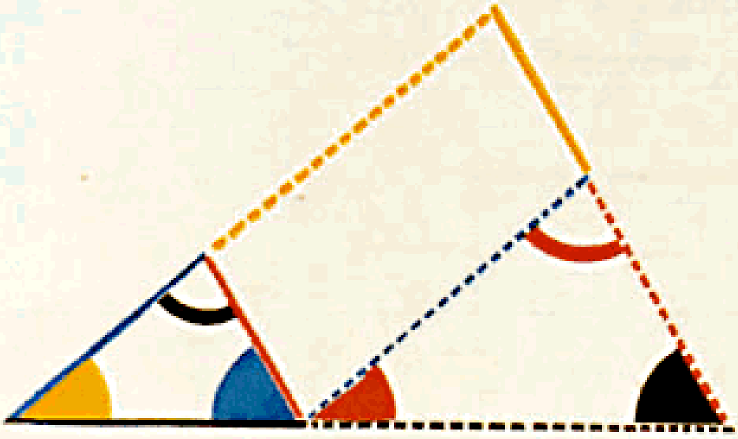
|| ; = ,

and = (B. 1. pr. 29);

∴ = , and = ,

and ∴ biſects .

Q. E. D.



IN equiangular triangles (and) the sides about the equal angles are proportional, and the sides which are opposite to the equal angles are homologous.

Let the equiangular triangles be so placed that two sides , opposite to equal angles and may be conterminous, and in the same straight line; and that the triangles lying at the same side of that straight line, may have the equal angles not conterminous,

i. e. opposite to , and to .

Draw and . Then, because

= , || (B. 1. pr. 28);

and for a like reason, || ,

∴ is a parallelogram.

But : :: : (B. 6. pr. 2);

and ſince  (B. 1. pr. 34),

 ; and by

alternation, 

(B. 5. pr. 16).

In like manner it may be ſhown, that

and by alternation, that

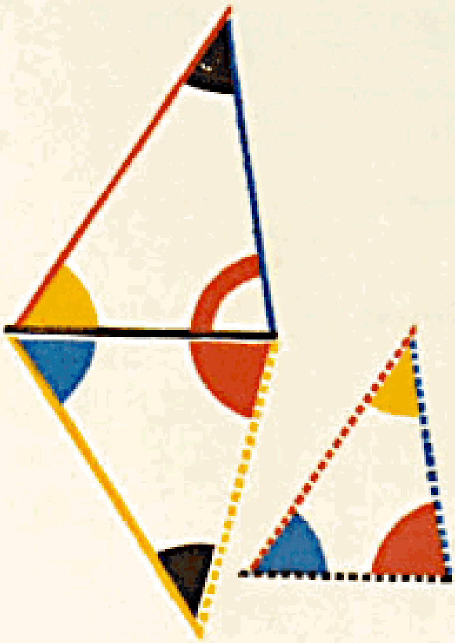
but it has been already proved that

and therefore, ex æquali,

(B. 5. pr. 22),

therefore the ſides about the equal angles are proportional, and thoſe which are oppoſite to the equal angles are homologous.

Q. E. D.



IF two triangles have their ſides proportional (: :: :) and (: :: :) *they are equiangular, and the equal angles are ſubtended by the homologous ſides.*

From the extremities of , draw and , making = , = (B. 1. pr. 23);

and conſequently = (B. 1. pr. 32), and ſince the triangles are equiangular,

: :: : (B. 6. pr. 4);

but : :: : (hyp.);

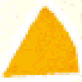
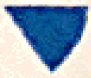
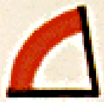
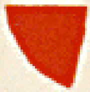
∴ : :: : ,

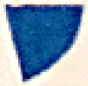
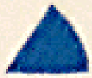
and conſequently = (B. 5. pr. 9).

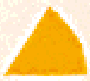
In the like manner it may be ſhown that

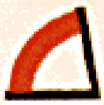
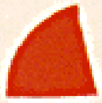
= .

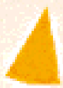
Therefore, the two triangles having a common baſe , and their ſides equal, have alſo equal angles oppoſite to equal ſides, i. e.

 =  and  =  (B. 1. pr. 8).

But  =  (conſt.)

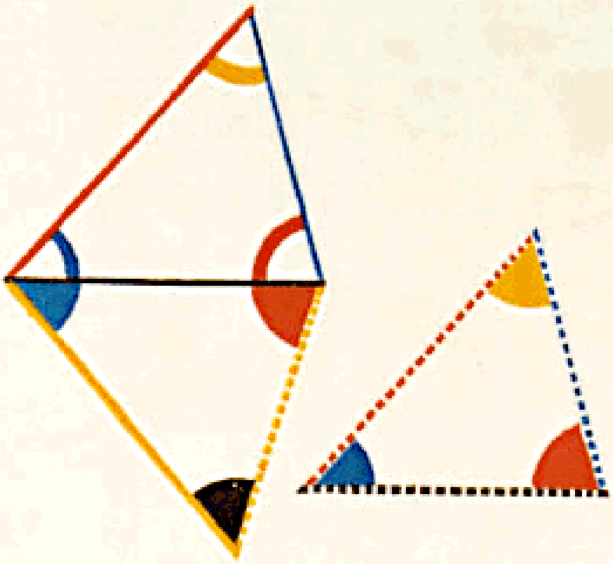
and ∴  = ; for the ſame

reaſon  = , and

conſequently  =  (B. 1. 32);

and therefore the triangles are equiangular, and it is evident that the homologous ſides ſubtend the equal angles.

Q. E. D.



IF two triangles (and) have one angle ( ) of the one, equal to one angle ( ) of the other, and the sides about the equal angles proportional, the triangles shall be equiangular, and have those angles equal which the homologous sides subtend.

From the extremities of , one of the sides of , about , draw and , making

= , and = ; then = (B. 1. pr. 32), and two triangles being equiangular,

: :: :

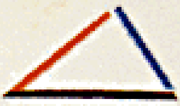
(B. 6. pr. 4);

but : :: : (hyp.);

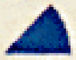
∴ : :: :

(B. 5. pr. 11),

and consequently = (B. 5. pr. 9);

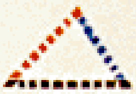
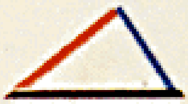
∴  =  in every reſpect.
(B. 1. pr. 4).

But  =  (conſt.),

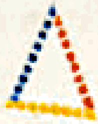
and ∴  = ; and

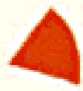
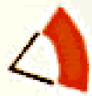
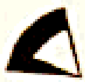
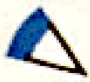
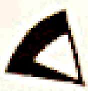
ſince alſo  = ,

 =  (B. 1. pr. 32);

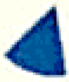
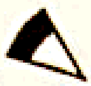
and ∴  and  are equiangular, with their equal angles oppoſite to homologous ſides.

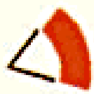
Q. E. D.

I*F two triangles (*  *and*  *) have one angle in each equal (*  *equal to*  *), the sides about two other angles proportional (*  *), and each of the remaining angles (*  *and*  *) either less or not less than a right angle, the triangles are equiangular, and those angles are equal about which the sides are proportional.*

First let it be assumed that the angles  and  are each less than a right angle: then if it be supposed that  and  contained by the proportional sides, are not equal, let  be the greater, and make

 = .

Because  =  (hyp.), and  =  (const.)

∴  =  (B. 1. pr. 32);

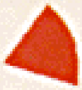
∴ : :: :

(B. 6. pr. 4),

but : :: : (hyp.)

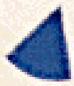
∴ : :: : ;

∴ = (B. 5. pr. 9),

and ∴  =  (B. 1. pr. 5).

But  is lefs than a right angle (hyp.)

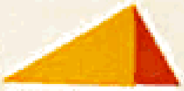
∴  is lefs than a right angle; and ∴  muft be greater than a right angle (B. 1. pr. 13), but it has been proved =  and therefore lefs than a right angle, which is abfurd. ∴  and  are not unequal;

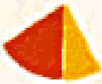
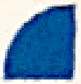
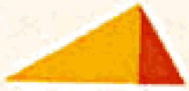
∴ they are equal, and fince  =  (hyp.)

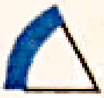
∴  =  (B. 1. pr. 32), and therefore the triangles are equiangular.

But if  and  be affumed to be each not lefs than a right angle, it may be proved as before, that the triangles are equiangular, and have the fides about the equal angles proportional. (B. 6. pr. 4).

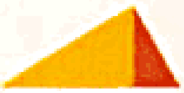
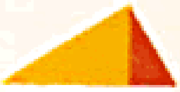
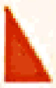
Q. E. D.

IN a right angled triangle (), *if a perpendicular* (——) *be drawn from the right angle to the oppoſite ſide, the triangles* ( , ) *on each ſide of it are ſimilar to the whole triangle and to each other.*

Becauſe  =  (B. 1. ax. 11), and  common to  and ;

 =  (B. 1. pr. 32);

∴  and  are equiangular; and conſequently have their ſides about the equal angles proportional (B. 6. pr. 4), and are therefore ſimilar (B. 6. def. 1).

In like manner it may be proved that  is ſimilar to ; but  has been ſhewn to be ſimilar to ; ∴  and  are ſimilar to the whole and to each other.

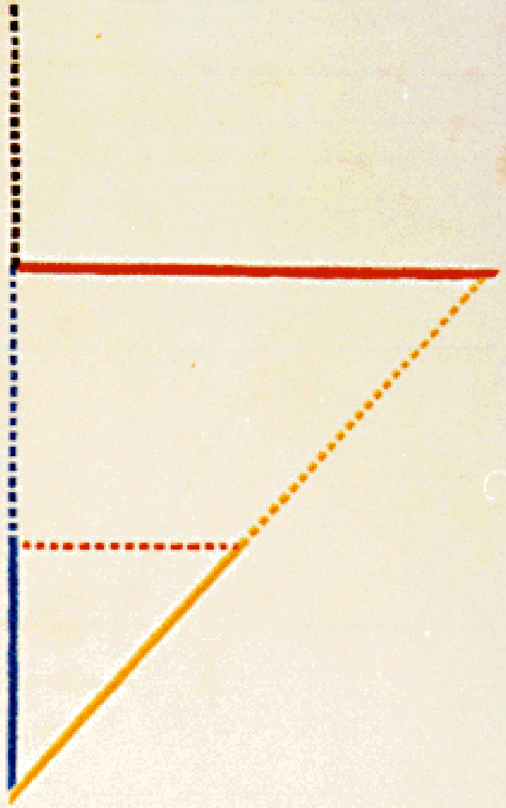
Q. E. D.

FROM *a given ſtraight line ( ) to cut off any required part.*

From either extremity of the given line draw making any angle with ; and produce till the whole produced line contains as often as contains the required part.

Draw , and draw || .

is the required part of .



For ſince ||

: :: :

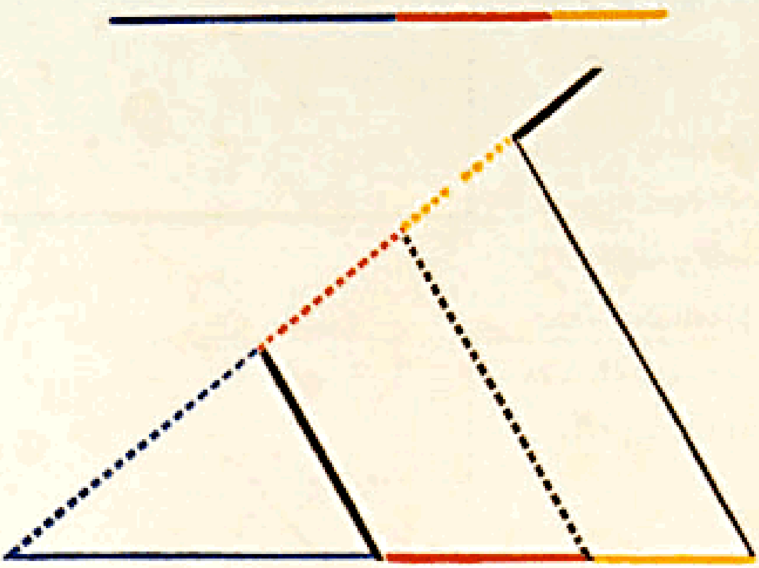
(B. 6. pr. 2), and by compoſition (B. 5. pr. 18);

: :: : ;

but contains as often as contains the required part (conſt.);

∴ is the required part.

Q. E. D.



TO divide a ſtraight line (———) *ſimilarly to a given divided line* (———).

From either extremity of the given line ——— draw ········· making any angle; take ·····, ····· and ····· equal to ———, ——— and ——— reſpectively (B. 1. pr. 2); draw ———, and draw ····· and ——— || to it.

Since { ····· ——— } are ||,

——— : ——— :: ····· : ·····

(B. 6. pr. 2),

or ——— : ——— :: ——— : ——— (conſt.),

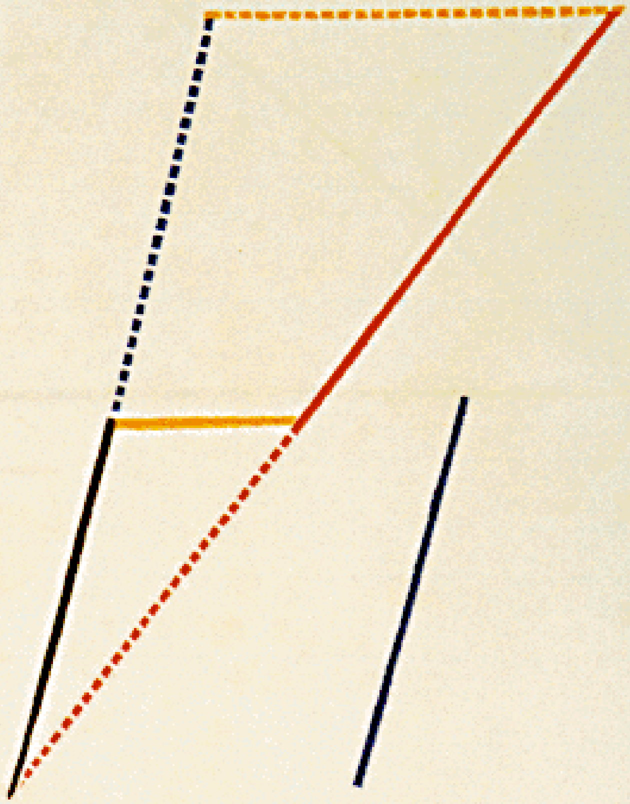
and ——— : ——— :: ····· : ·····

(B. 6. pr. 2),

——— : ——— :: ——— : ——— (conſt.),

and ∴ the given line ——— is divided ſimilarly to ———.

Q. E. D.



TO *find a third proportional to two given straight lines* (*and*).

At either extremity of the given line draw making an angle; take = , and draw ; make = , and draw || ; (B. 1. pr. 31.) is the third proportional to and .

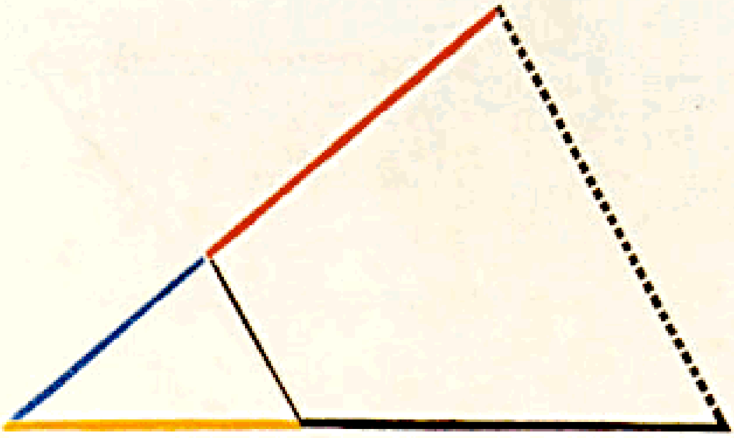
For since || ,

∴ : :: : (B. 6 pr. 2);

but = = (conſt.);

∴ : :: : . (B. 5. pr. 7).

Q. E. D.



To find a fourth proportional to three given lines

{ }.

Draw

and making any angle;

take = ,

and = ,

alfo = ,

draw ,

and || ;

(B. 1. pr. 31);

is the fourth proportional.

On account of the parallels,

: :: :

(B. 6. pr. 2);

but { } = { } (conft.);

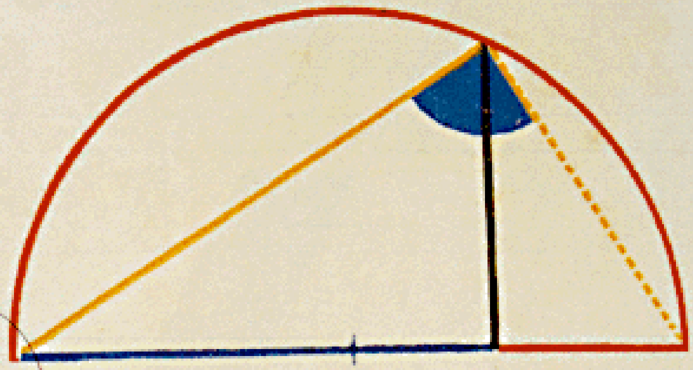
∴ : :: : .

(B. 5. pr. 7).

Q. E. D.

T*O find a mean proportional between two given ſtraight lines*

{ }.



Draw any ſtraight line ,

make = ,

and = ; biſect :

and from the point of biſection as a centre, and half the

line as a radius, deſcribe a ſemicircle  ,

draw ⊥ :

is the mean proportional required.

Draw and .

Since  is a right angle (B. 3. pr. 31),

and is ⊥ from it upon the oppoſite ſide,

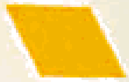
∴ is a mean proportional between

and (B. 6. pr. 8),

and ∴ between and (conſt.).

Q. E. D

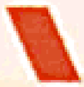
I.

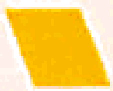
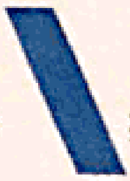
EQUAL *parallelograms*  *and* , *which have one angle in each equal, have the ſides about the equal angles reciprocally proportional*
(: :: :).

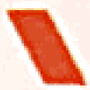
II.

And parallelograms which have one angle in each equal, and the ſides about them reciprocally proportional, are equal.

Let and ; and and , be ſo placed that and may be continued right lines. It is evident that they may aſſume this poſition. (B. 1. prs. 13, 14, 15.)

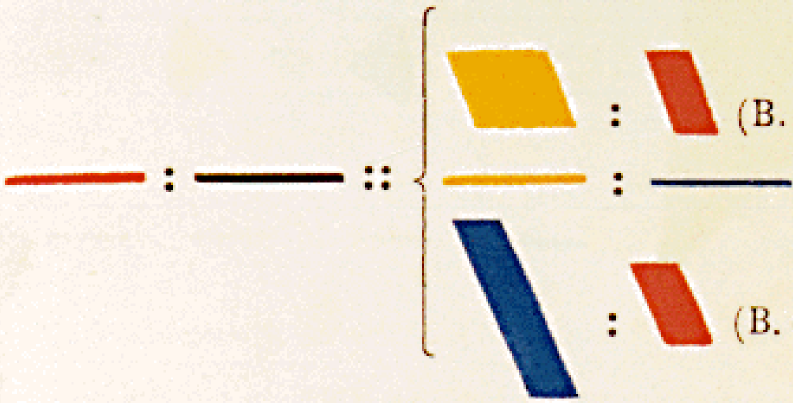
Complete .

Since  = ;

∴  :  ::  :  (B. 5. pr. 7.)

(B. 6. pr. 1.)

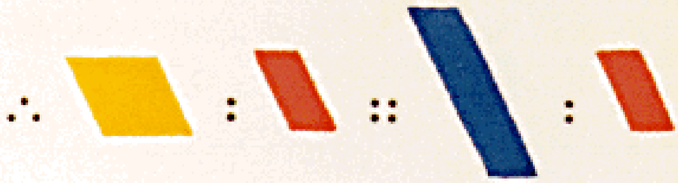
The ſame conſtruction remaining :



(B. 6. pr. 1.)

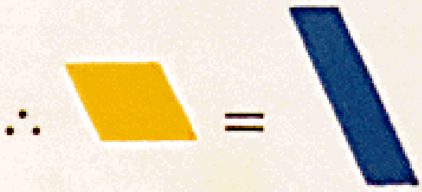
(hyp.)

(B. 6. pr. 1.)



(B. 5. pr. 11.)

and



(B. 5. pr. 9).

Q. E. D.

I.

EQUAL *triangles, which have one angle in each equal* (=), *have the ſides about the equal angles reciprocally proportional*
(: :: :).

II.

And two triangles which have an angle of the one equal to an angle of the other, and the ſides about the equal angles reciprocally proportional, are equal.

I.

Let the triangles be ſo placed that the equal angles and may be vertically oppoſite, that is to ſay, ſo that and may be in the ſame ſtraight line. Whence alſo and muſt be in the ſame ſtraight line. (B. 1. pr. 14.)

Draw , then

: :: : (B. 6. pr. 1.)

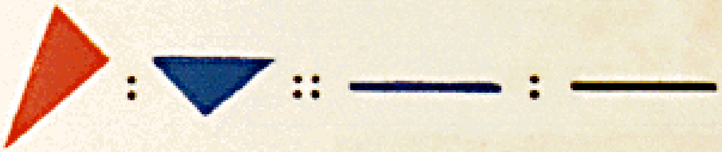
:: : (B. 5. pr. 7.)

:: : (B. 6. pr. 1.)

(B. 5. pr. 11.)

II.

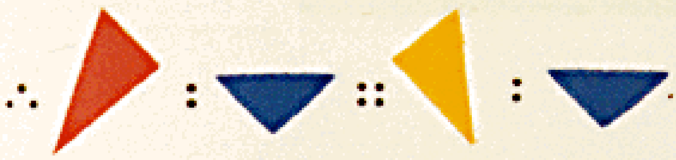
Let the fame conftruction remain, and

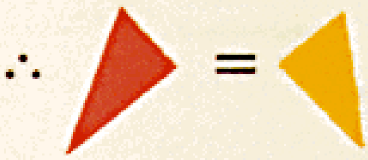
 (B. 6. pr. 1.)

and 

(B. 6. pr. 1.)

But , (hyp.)

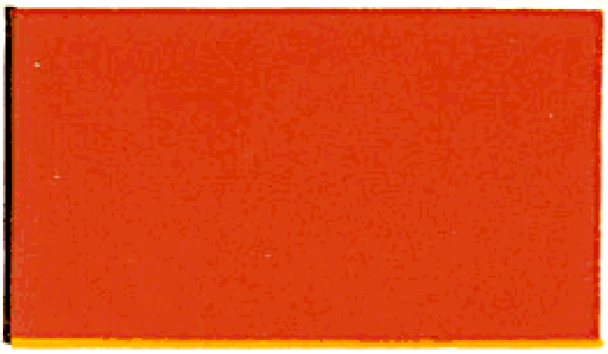
. (B. 5 pr. 11);

 (B. 5. pr. 9.)

Q. E. D.

PART I.

IF four straight lines be proportional (—— : —— :: ----- : -----), the rectangle (—— × -----) contained by the extremes, is equal to the rectangle (—— × -----) contained by the means.



PART II.

And if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four straight lines are proportional.

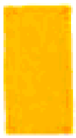
PART I.

From the extremities of —— and —— draw —— and —— ⊥ to them and = ----- and ----- respectively: complete the parallelograms  and .

And since,

—— : —— :: ----- : ----- (hyp.)

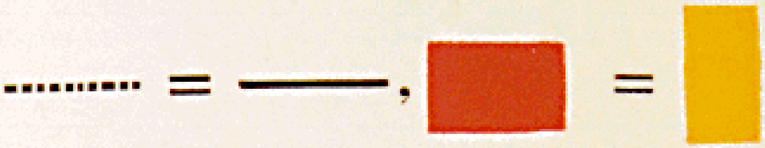
∴ —— : —— :: —— : —— (const.)

∴  =  (B. 6. pr. 14).

that is, the rectangle contained by the extremes, equal to the rectangle contained by the means.

PART II.

Let the ſame conſtruction remain; becauſe



and 

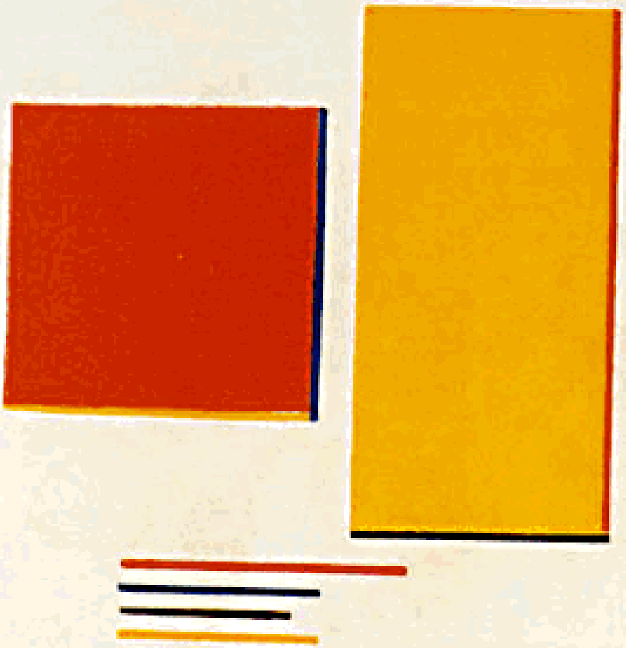
(B. 6. pr. 14).

But 

and  (conſt.)

(B. 5. pr. 7).

Q. E. D.



PART I

If three ſtraight lines be proportional (—— : —— :: —— : ——) the rectangle under the extremes is equal to the ſquare of the mean.

PART II.

And if the rectangle under the extremes be equal to the ſquare of the mean, the three ſtraight lines are proportional.

PART I.

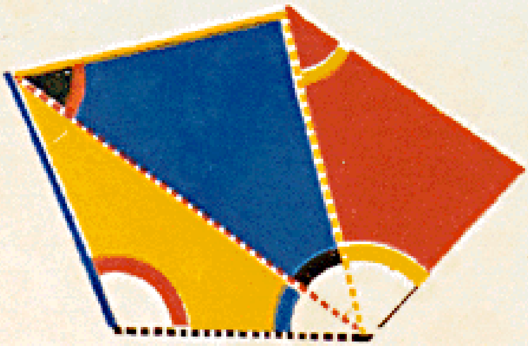
Aſſume —— = ——, and
ſince —— : —— :: —— : ——,
then —— : —— :: —— : ——,
∴ —— × —— = —— × ——
(B. 6. pr. 16).

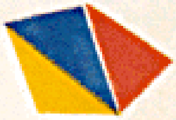
But —— = ——,
∴ —— × —— = —— × ——,
or = ——²; therefore, if the three ſtraight lines are proportional, the rectangle contained by the extremes is equal to the ſquare of the mean.

PART II.

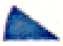
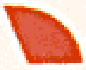
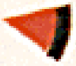
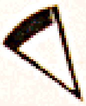
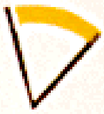
Aſſume —— = ——, then
—— × —— = —— × ——,
∴ —— : —— :: —— : ——
(B. 6. pr. 16), and
∴ —— : —— :: —— : ——.

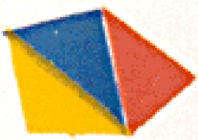
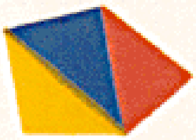
Q. E. D.

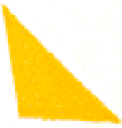
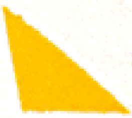


ON a given straight line (———) *to construct a rectilinear figure similar to a given one* () *and similarly placed.*

Resolve the given figure into triangles by drawing the lines - - - - and - - - -.

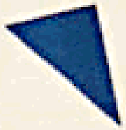
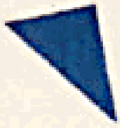
At the extremities of ——— make  =  and  = : again at the extremities of ——— make  =  and  =  : in like manner make  =  and  = .

Then  is similar to .

It is evident from the construction and (B. 1. pr. 32) that the figures are equiangular; and since the triangles  and  are equiangular; then by (B. 6. pr. 4),

——— : ——— :: - - - - : ———

and ——— : ——— :: ——— : - - - -.

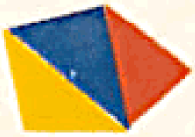
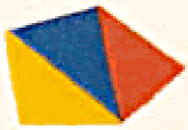
Again, because  and  are equiangular,

∴ ex æquali,

(B. 6. pr. 22.)

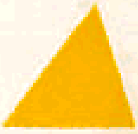
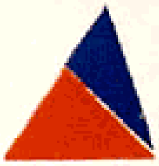
In like manner it may be shown that the remaining sides of the two figures are proportional.

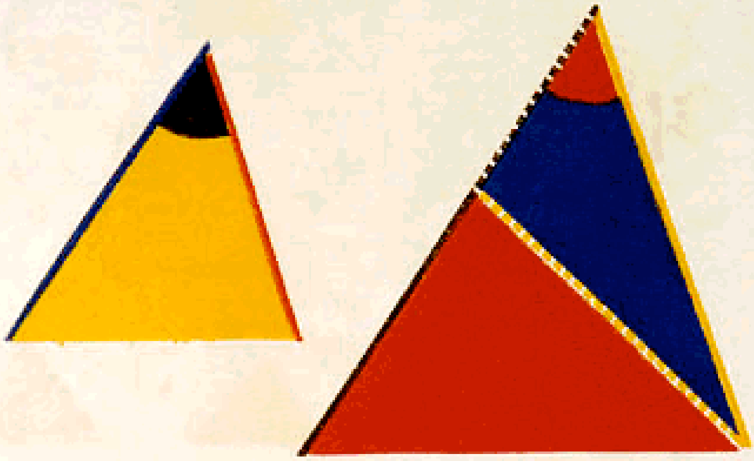
∴ by (B. 6. def. 1.)

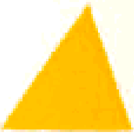
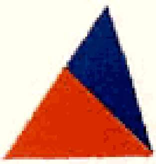
 is similar to 

and similarly situated; and on the given line ———.

Q. E. D.

SIMILAR *triangles* ( *and* ) *are to one another in the duplicate ratio of their homologous sides.*



Let  and  be equal angles, and and homologous sides of the similar triangles  and  and on the greater of these lines take a third proportional, so that

 : :: : ;

draw .

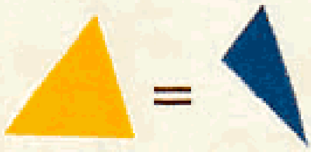
 : :: :

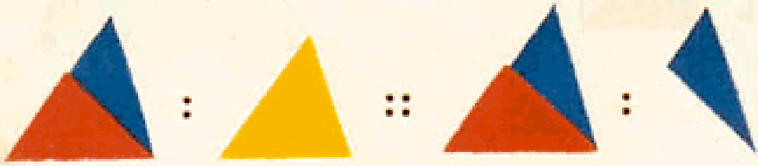
(B. 6. pr. 4);

∴ : :: :

(B. 5. pr. 16, alt.),

but :  ::  : (const.),

∴ : :: : consequently  for they have the ſides about the equal angles  and  reciprocally proportional (B. 6. pr. 15);

∴ 

(B. 5 pr. 7);

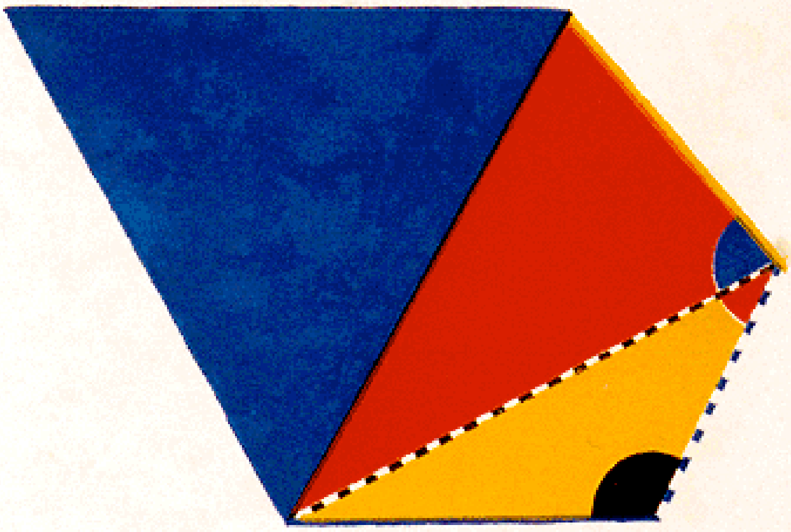
but 

(B. 6. pr. 1),

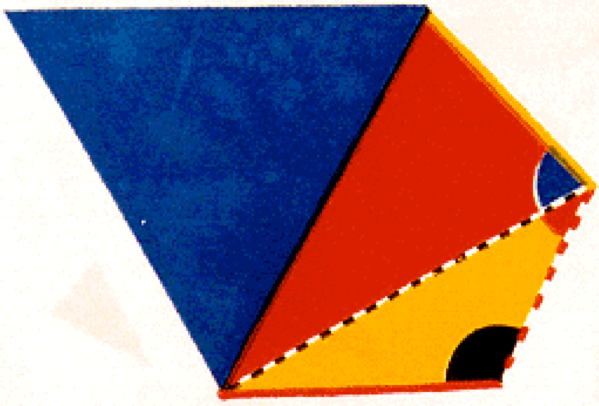
∴ 

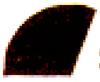
that is to ſay, the triangles are to one another in the duplicate ratio of their homologous ſides  and (B. 5. def. 11).

Q. E. D.

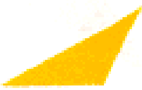
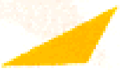
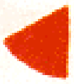


SIMILAR *polygons may be divided into the same number of similar triangles, each similar pair of which are proportional to the polygons; and the polygons are to each other in the duplicate ratio of their homologous sides.*

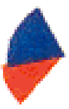
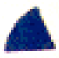


Draw and , and and , resolving the polygons into triangles. Then because the polygons are similar,  = ,

and : :: :

∴  and  are similar, and  = 

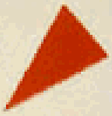
(B. 6. pr. 6);

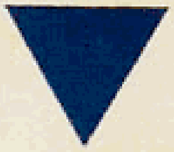
but  =  because they are angles of similar polygons; therefore the remainders  and  are equal;

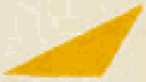
hence : :: : ,

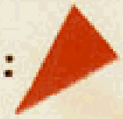
on account of the similar triangles,

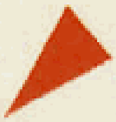
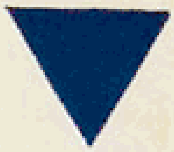
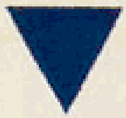
and : :: : ,
on account of the ſimilar polygons,

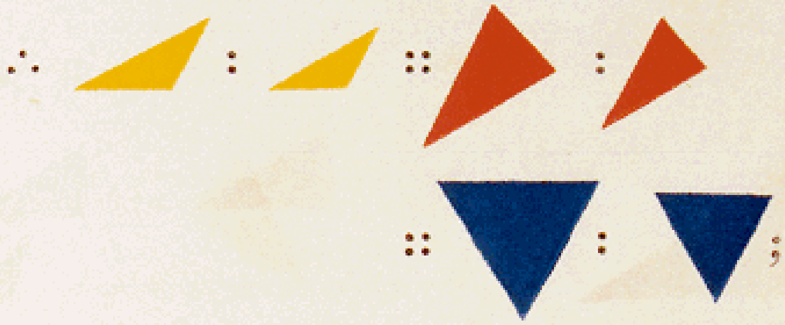
∴ : :: : ,
ex æquali (B. 5. pr. 22), and as theſe proportional ſides contain equal angles, the triangles  and  are ſimilar (B. 6. pr. 6).

In like manner it may be ſhown that the triangles  and  are ſimilar.

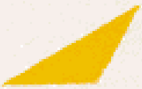
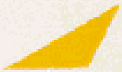
But  is to  in the duplicate ratio of to (B. 6. pr. 19), and  is to  in like manner, in the duplicate ratio of to ;

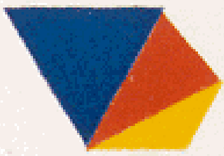
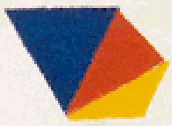
∴  :  ::  :  .
(B. 5. pr. 11);

Again  is to  in the duplicate ratio of to , and  is to  in the duplicate ratio of —— to ——,

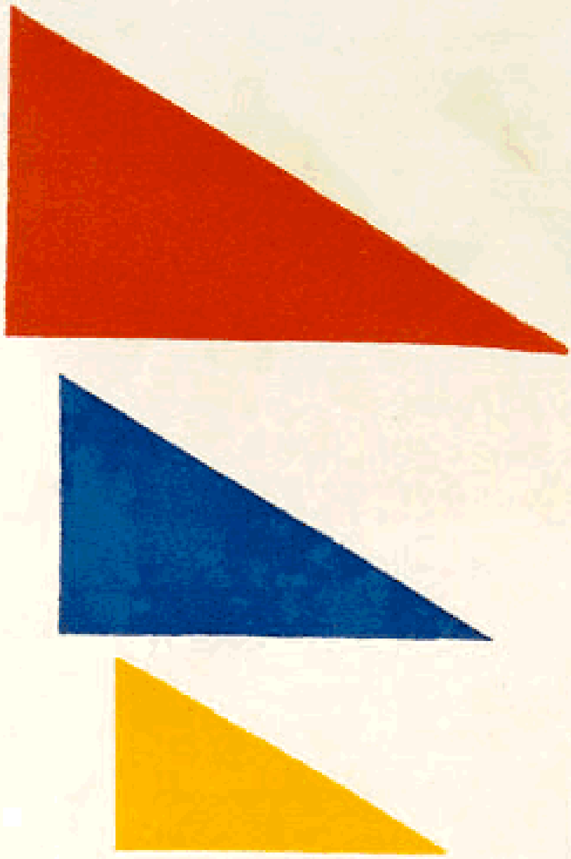


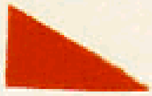
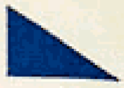
and as one of the antecedents is to one of the consequents, so is the sum of all the antecedents to the sum of all the consequents; that is to say, the similar triangles have to one another the same ratio as the polygons (B. 5. pr. 12).

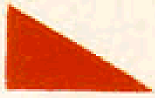
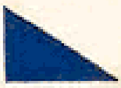
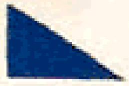
But  is to  in the duplicate ratio of —— to ——;

∴  is to  in the duplicate ratio of —— to ——.

Q. E. D



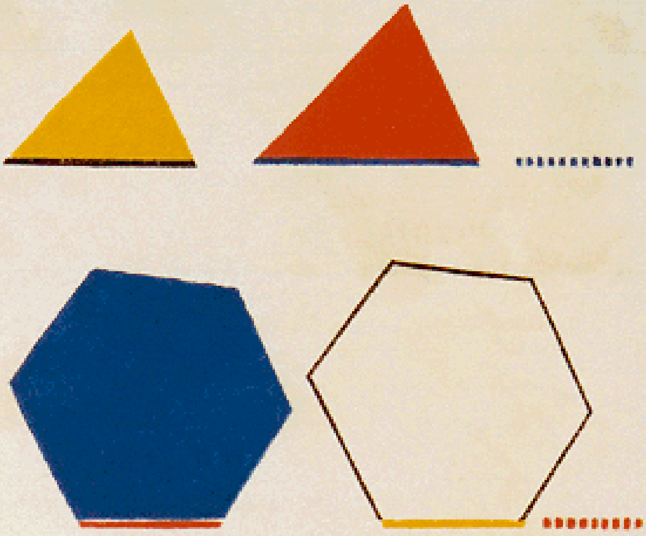
RECTILINEAR *figures* ( *and* ) *which are similar to the same figure* () *are similar also to each other.*

Since  and  are similar, they are equiangular, and have the sides about the equal angles proportional (B. 6. def. 1); and since the figures  and  are also similar, they are equiangular, and have the sides about the equal angles proportional; therefore  and  are also equiangular, and have the sides about the equal angles proportional (B. 5. pr. 11), and are therefore similar.

Q. E. D.

PART I.

IF four straight lines be proportional (: :: :), the similar rectilinear figures similarly described on them are also proportional.



PART II.

And if four similar rectilinear figures, similarly described on four straight lines, be proportional, the straight lines are also proportional.

PART I.

Take a third proportional to and , and a third proportional to and (B. 6. pr. 11);

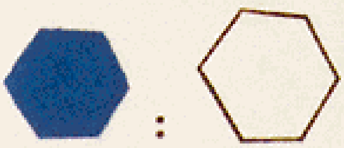
since : :: : (hyp.),

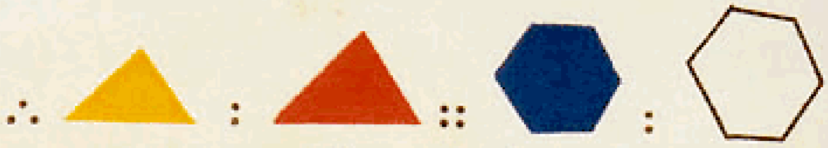
: :: : (const.)

∴ ex æquali,

: :: : ;

but  :: : (B. 6. pr. 20),

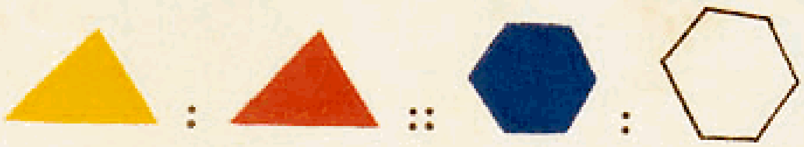
and  :: : ;



(B. 5. pr. 11).

PART II.

Let the ſame conſtruction remain :

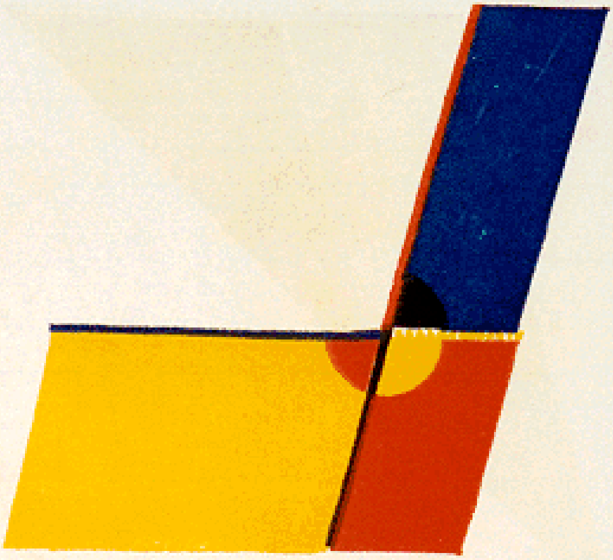


(hyp.),

(B. 5. pr. 11).

Q. E. D.

EQUIANGULAR *parallelograms (and) are to one another in a ratio compounded of the ratios of their ſides.*



Let two of the ſides and about the equal angles be placed ſo that they may form one ſtraight line.

Since + = ,

and = (hyp.),

+ = ,

and ∴ and form one ſtraight line (B. 1. pr. 14);

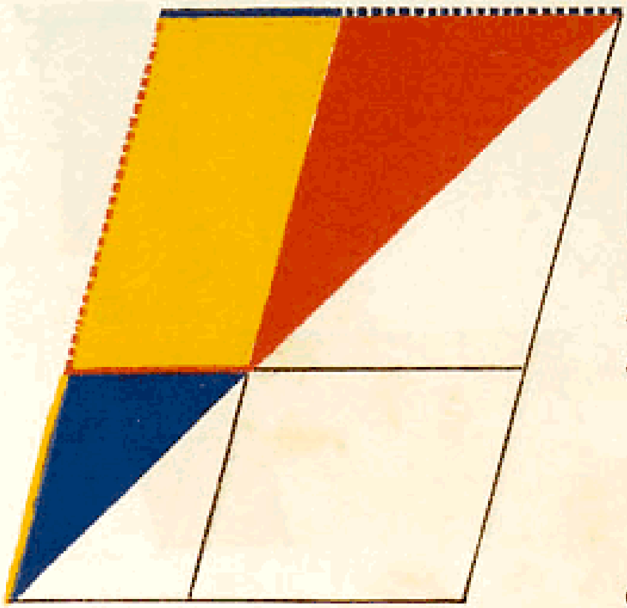
complete .

Since : :: : (B. 6. pr. 1),

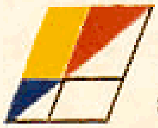
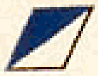
and : :: : (B. 6. pr. 1),

has to a ratio compounded of the ratios of to , and of to .

 Q. E. D.



IN any parallelogram () *the parallelograms* ( *and* ) *which are about the diagonal are ſimilar to the whole, and to each other.*

As  and  have a common angle they are equiangular;

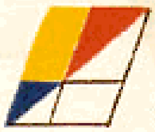
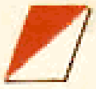
but becauſe ||

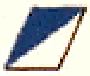
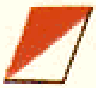
 and  are ſimilar (B. 6. pr. 4),

∴ : ::  : ;

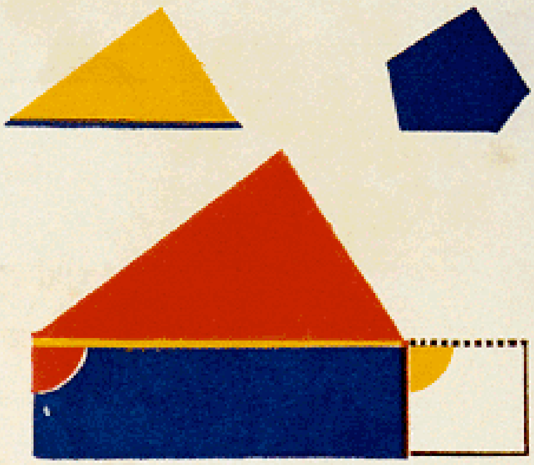
and the remaining oppoſite ſides are equal to thoſe,

∴  and  have the ſides about the equal angles proportional, and are therefore ſimilar.

In the ſame manner it can be demonſtrated that the parallelograms  and  are ſimilar.

Since, therefore, each of the parallelograms  and  is ſimilar to , they are ſimilar to each other.

Q. E. D.



TO describe a rectilinear figure, which shall be similar to a given rectilinear figure ( ), *and equal to another* ( ).

Upon describe = ,

and upon describe = ,

and having = (B. 1. pr. 45), and then

and will lie in the same straight line

(B. 1. prs. 29, 14),

Between and find a mean proportional

(B. 6. pr. 13), and upon

describe , similar to ,

and similarly situated.

Then = .

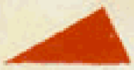
For since and are similar, and

: :: : (const.),

: :: :

(B. 6. pr. 20);

but  :  :: : (B. 6. pr. 1);

∴  :  ::  :  (B. 5. pr. 11);

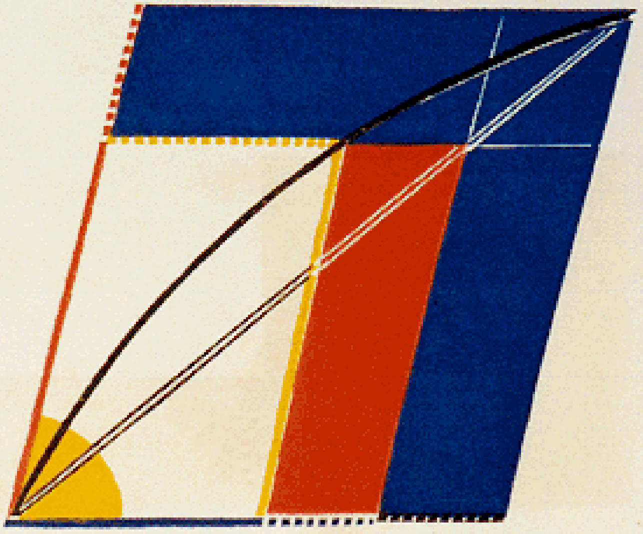
but  =  (conſt.),

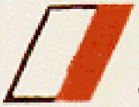
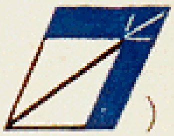
and ∴  =  (B. 5. pr. 14);

and  =  (conſt.); conſequently,

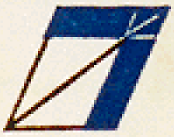
 which is ſimilar to  is alſo = .

Q. E. D.

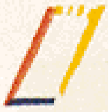
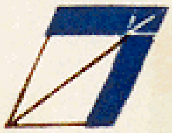


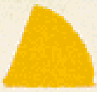
If similar and similarly posited parallelograms ( *and* ) *have a common angle, they are about the same diagonal.*

For, if possible, let

be the diagonal of  and

draw || (B. 1. pr. 31).

Since  and  are about the same

diagonal , and have  common,

they are similar (B. 6. pr. 24);

∴ : :: : ;

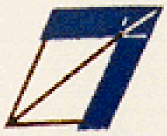
but : :: :

(hyp.),

∴ : :: : ,

and ∴ = (B. 5. pr. 9.),

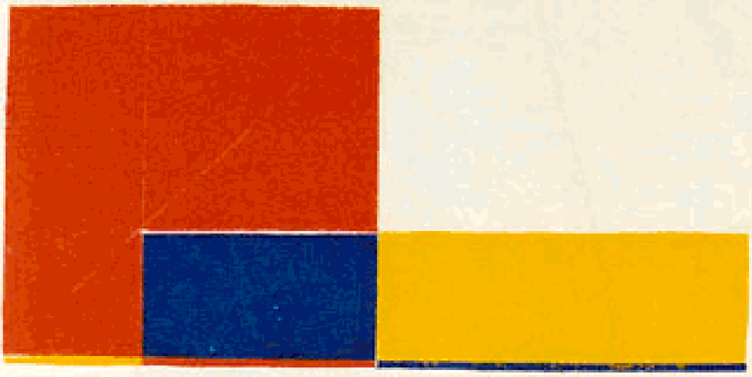
which is absurd.

∴ is not the diagonal of 

in the same manner it can be demonstrated that no other

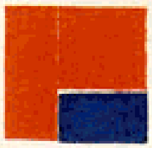
line is except .

Q. E. D.



OF all the rectangles contained by the ſegments of a given ſtraight line, the greateſt is the ſquare which is deſcribed on half the line.

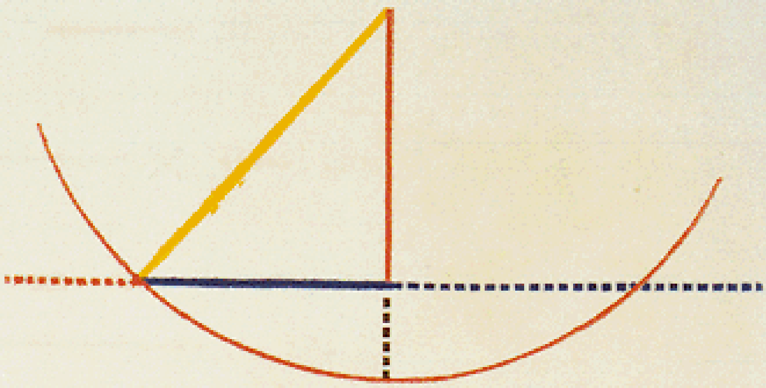
Let be the given line, and unequal ſegments, and and equal ſegments;

then  ⊏ .

For it has been demonſtrated already (B. 2. pr. 5), that the ſquare of half the line is equal to the rectangle contained by any unequal ſegments together with the ſquare of the part intermediate between the middle point and the point of unequal ſection. The ſquare deſcribed on half the line exceeds therefore the rectangle contained by any unequal ſegments of the line.

Q. E. D.

TO divide a given straight line ( ) so that the rectangle contained by its segments may be equal to a given area, not exceeding the square of half the line.



Let the given area be = ².

Bisect , or

make = ;

and if ² = ²,

the problem is solved.

But if ² ≠ ², then

must ⊏ (hyp.).

Draw ⊥ = ;

make = or ;

with as radius describe a circle cutting the given line; draw .

Then × + ² = ²

(B. 2. pr. 5.) = ².

But ² = ² + ²

(B. 1. pr. 47);

∴ × + ²

= ² + ²,

from both, take ²,

and × = ².

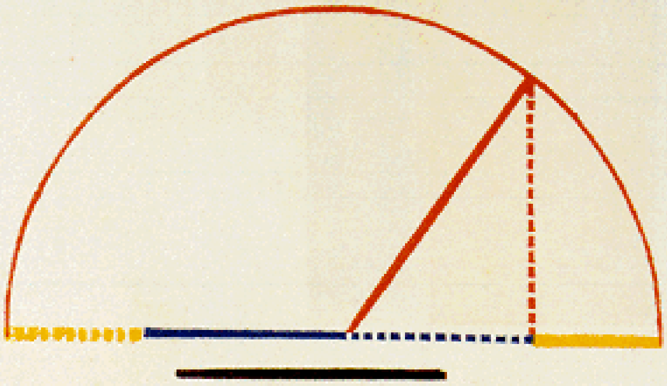
But = (conſt.),

and ∴ is ſo divided

that × = ².

Q. E. D.

TO produce a given straight line (———), so that the rectangle contained by the segments between the extremities of the given line and the point to which it is produced, may be equal to a given area, i. e. equal to the square on ———.



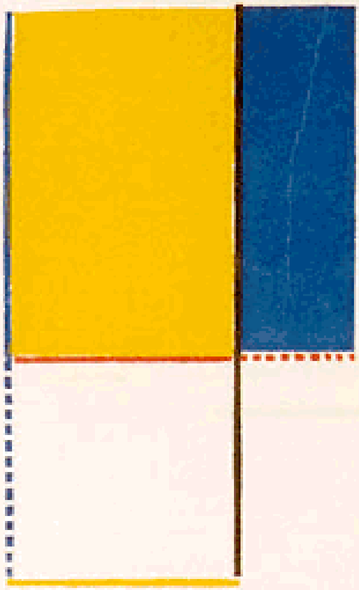
Make ——— = ———, and
draw ——— ⊥ ——— = ———;
draw ———; and
with the radius ———, describe a circle
meeting ——— produced.

Then ——— × ——— + ———^{2} =
———^{2} (B. 2. pr. 6.) = ———^{2}.

But ———^{2} = ———^{2} + ———^{2} (B. 1. pr. 47.)

∴ ——— × ——— + ———^{2} =
———^{2} + ———^{2},
from both take ———^{2},
and ——— × ——— = ———^{2};
but ——— = ———,
∴ ———^{2} = the given area.

Q. E. D.



TO cut a given finite ſtraight line ( ) in extreme and mean ratio.

On describe the ſquare (B. 1. pr. 46); and produce , ſo that × = ² (B. 6. pr. 29);

take = ,

and draw ∥ ,

meeting ∥ (B. 1. pr. 31).

Then = × ,

and is ∴ = ; and if from both theſe equals

be taken the common part ,

, which is the ſquare of ,

will be = , which is = × ;

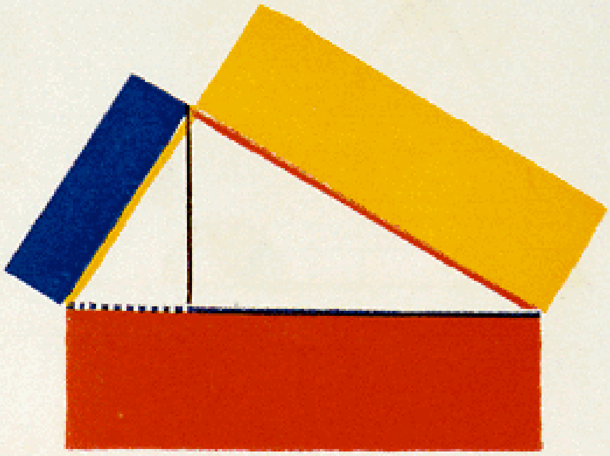
that is ² = × ;

∴ : :: : ,

and is divided in extreme and mean ratio.

(B. 6. def. 3).

Q. E. D.



IF any similar rectilinear figures be similarly described on the sides of a right angled triangle ( ), the figure described on the side ( ) subtending the right angle is equal to the sum of the figures on the other sides.

From the right angle draw perpendicular to ;

then : :: :

(B. 6. pr. 8).

∴ : :: :

(B. 6. pr. 20).

but : :: :

(B. 6. pr. 20).

Hence + :

:: + : ;

but + = ;

and ∴ + = .

Q. E. D.

IF two triangles (and), have two ſides proportional (: :: :), and be ſo placed at an angle that the homologous ſides are parallel, the remaining ſides (and) form one right line.

Since || ,

= (B. 1. pr. 29);

and alſo ſince || ,

= (B. 1. pr. 29);

∴ = ; and ſince

: :: : (hyp.),

the triangles are equiangular (B. 6. pr. 6);

∴ = :

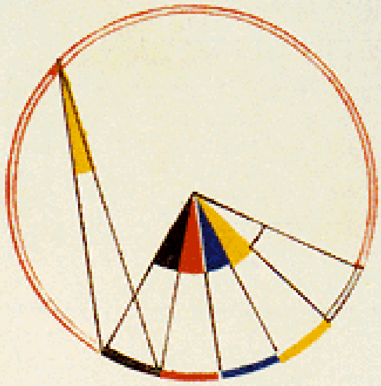
but = ;

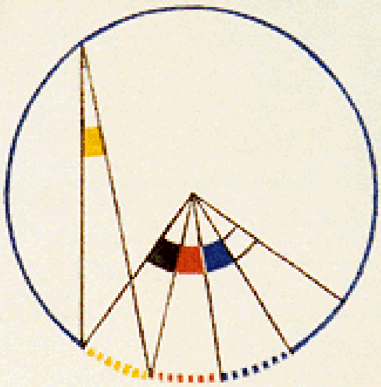
∴ + + = + + =

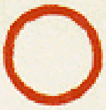
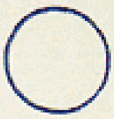
(B. 1. pr. 32), and ∴ and

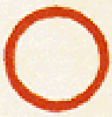
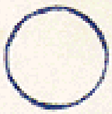
lie in the ſame ſtraight line (B. 1. pr. 14).

Q. E. D.

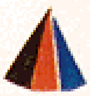




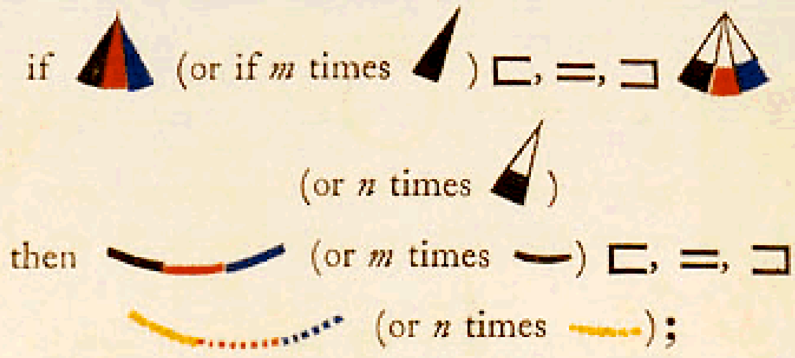
IN *equal circles* ( , ), *angles, whether at the centre or circumference, are in the ſame ratio to one another as the arcs on which they ſtand* ( :  ::  : ); *ſo alſo are ſectors.*

Take in the circumference of  any number of arcs , , &c. each =  , and alſo in the circumference of  take any number of arcs ,  , &c. each =  , draw the radii to the extremities of the equal arcs.

Then ſince the arcs , ,  , &c. are all equal, the angles  ,  ,  , &c. are alſo equal (B. 3. pr. 27);

∴  is the ſame multiple of  which the arc  is of  ; and in the ſame manner  is the ſame multiple of  , which the arc  is of the arc .

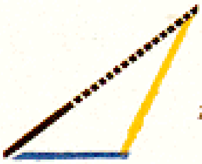
Then it is evident (B. 3. pr. 27),

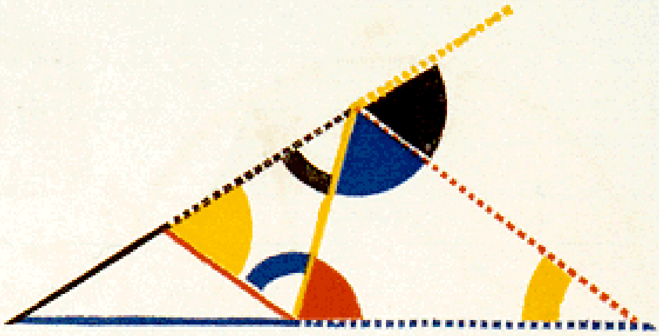


∴ : :: : , (B. 5. def. 5), or the angles at the centre are as the arcs on which they ſtand; but the angles at the circumference being halves of the angles at the centre (B. 3. pr. 20) are in the ſame ratio (B. 5. pr. 15), and therefore are as the arcs on which they ſtand.

It is evident, that ſectors in equal circles, and on equal arcs are equal (B. 1. pr. 4; B. 3. prs. 24, 27, and def. 9). Hence, if the ſectors be ſubſtituted for the angles in the above demonſtration, the ſecond part of the propoſition will be eſtabliſhed, that is, in equal circles the ſectors have the ſame ratio to one another as the arcs on which they ſtand.

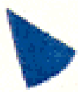
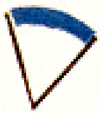
Q. E. D.

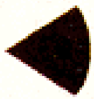
If the right line (), *bisecting an external angle*  *of the triangle*  *meet the opposite*

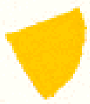


side () *produced, that whole produced side* (), *and its external segment* () *will be proportional to the sides* (*and* ), *which contain the angle adjacent to the external bisected angle.*

For if be drawn ∥ ,

then  = , (B. 1. pr. 29);

= , (hyp.),

= , (B. 1. pr. 29);

and ∴ = , (B. 1. pr. 6),

and : :: : ,

(B. 5. pr. 7);

But also,

 : :: : ,

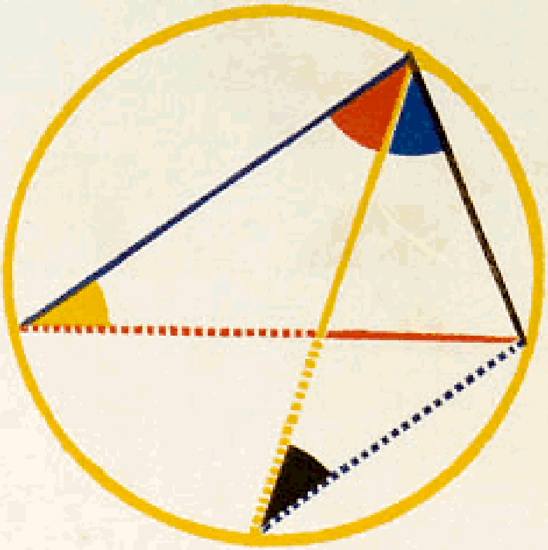
(B. 6. pr. 2);

and therefore

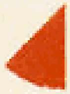
 : :: : ,

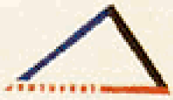
(B. 5. pr. 11).

Q. E. D.



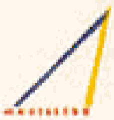
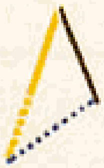
IF an angle of a triangle be bisected by a straight line, which likewise cuts the base; the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base, together with the square of the straight line which bisects the angle.

Let be drawn, making  = ; then shall

About  describe  (B. 4. pr. 5), produce to meet the circle, and draw .

Since  =  (hyp.),

and  =  (B. 3. pr. 21),

∴  and  are equiangular (B. 1. pr. 32);

∴ 

(B. 6. pr. 4);

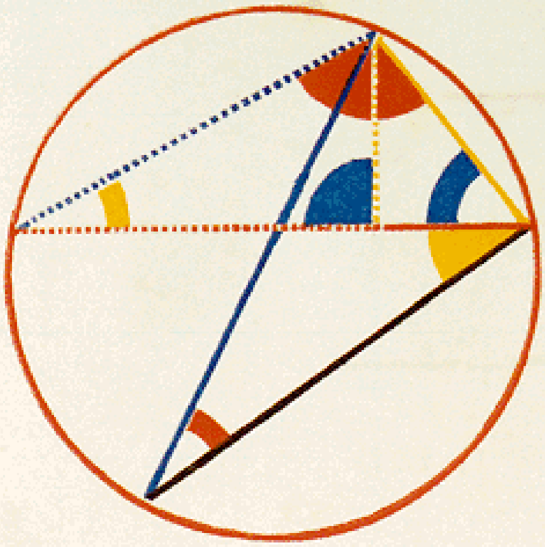
(B. 6. pr. 16.)

(B. 2. pr. 3);

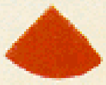
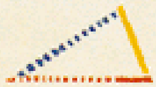
but 

(B. 3. pr. 35);

Q. E. D.

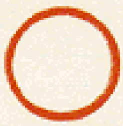


IF from any angle of a triangle a ſtraight line be drawn perpendicular to the baſe; the rectangle contained by the ſides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circle deſcribed about the triangle.

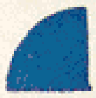
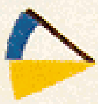
From  of 

draw ⊥ ; then

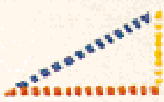
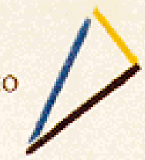
ſhall × = × the diameter of the deſcribed circle.

Deſcribe  (B. 4. pr. 5), draw its diameter

, and draw ; then becauſe

 =  (conſt. and B. 3. pr. 31);

and  =  (B. 3. pr. 21);

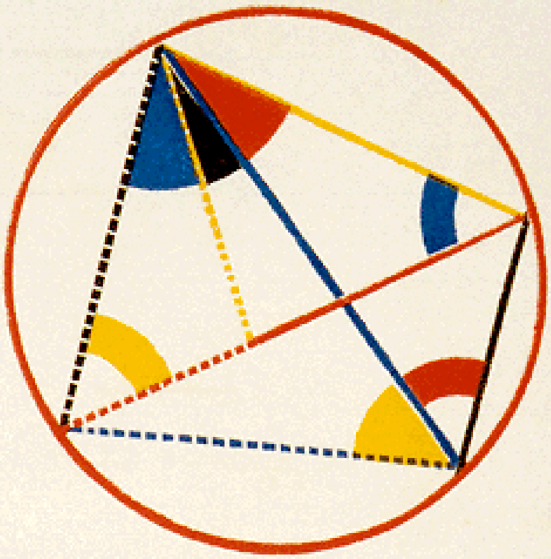
∴  is equiangular to  (B. 6. pr. 4);

∴  : :: : ;

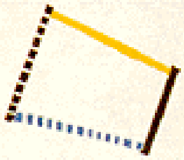
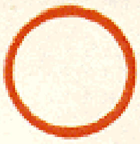
and ∴ × = ×

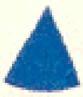
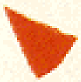
(B. 6. pr. 16).

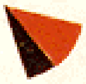
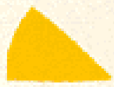
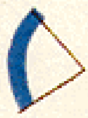
Q. E. D.



THE *rectangle contained by the diagonals of a quadrilateral figure inſcribed in a circle, is equal to both the rectangles contained by its oppoſite ſides.*

Let  be any quadrilateral figure inſcribed in ; and draw and ; then

Make  =  (B. 1. pr. 23),

∴  = ; and  = 

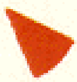
(B. 3. pr. 21);

∴ 

(B. 6. pr. 4);

and ∴ 

(B. 6. pr. 16); again,

becauſe  =  (conſt.), and  (B. 3. pr. 21);

(B. 6. pr. 4);

and 

(B. 6. pr. 16);

but, from above,

(B. 2. pr. 1.

Q. E. D.

THE END.

CHISWICK: PRINTED BY C. WHITTINGHAM.